Lecture 14 – 1st law for open systems

Purdue ME 200, Thermodynamics I

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Outline

Energy flow due to mass flow

1st law for open systems

Example: the Betz limit

Notation reminders

- in these notes,
  - ◇ V denotes volume (m^3)
  - ◇ ρ denotes density (kg/m^3)
  - ◇ p denotes pressure (Pa)
  - ◇ $v = V/m = 1/\rho$ denotes specific volume (m^3/kg)
  - ◇ $\dot{r}$ denotes the magnitude of velocity (m/s)
- caution:
  - ◇ the textbook uses V for both volume and velocity
  - ◇ don't mix up ρ and p!

1st law for closed systems

- the 1st law for closed systems can be written as

$$\Delta E = Q - W$$

- $E = \text{KE} + \text{PE} + U$ is total energy
- W includes boundary work, shaft work, electrical work, ...
- in rate form, 1st law is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W}$$

- today, we'll make 2 modifications to account for mass flow

Modification #1: mass flows carry energy

- mass that enters an open system brings energy with it
- this can be kinetic, potential or internal energy
- in general, mass inflow $\dot{m}^{\text{in}}$ carries total energy inflow

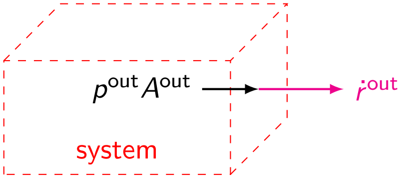
$$\begin{aligned}\dot{E}^{\text{in}} &= \frac{1}{2}\dot{m}^{\text{in}}(\dot{r}^{\text{in}})^2 + \dot{m}^{\text{in}} g z^{\text{in}} + \dot{U}^{\text{in}} \\ &= \dot{m}^{\text{in}} e^{\text{in}}\end{aligned}$$

- e^{in} is specific total energy of entering mass,

$$e^{\text{in}} = \frac{1}{2}(\dot{r}^{\text{in}})^2 + g z^{\text{in}} + u^{\text{in}}$$

- similarly, mass outflow $\dot{m}^{\text{out}}$ carries $\dot{m}^{\text{out}} e^{\text{out}}$ out of system

Modification #2: flow power



- suppose $\dot{m}^{out}$ leaves system in one-dimensional flow
- pressure at outlet surface (which has area A^{out}) is p^{out}
- as mass leaves, system pushes on it with force $p^{out}A^{out}$
- recall that
  - a force acting over a displacement does work
  - the time-derivative of work is power
  - power is force times velocity
- so system exerts **flow power** $p^{out}A^{out}\dot{r}^{out}$ on mass

Modification #2: flow power (continued)

- $\dot{V}^{\text{out}} = A^{\text{out}} \dot{r}^{\text{out}}$ is volumetric flow rate (m^3/s)
- volumetric and mass flow rates are related by

$$\dot{m}^{\text{out}} = \rho^{\text{out}} \dot{V}^{\text{out}} = \frac{\dot{V}^{\text{out}}}{v^{\text{out}}} = \frac{A^{\text{out}} \dot{r}^{\text{out}}}{v^{\text{out}}}$$

- so flow power can be written as

$$p^{\text{out}} A^{\text{out}} \dot{r}^{\text{out}} = p^{\text{out}} \dot{m}^{\text{out}} v^{\text{out}}$$

- in summary,
  - ◇ system exerts power $p^{\text{out}} \dot{m}^{\text{out}} v^{\text{out}}$ on mass outflow $\dot{m}^{\text{out}}$
  - ◇ mass inflow $\dot{m}^{\text{in}}$ exerts power $p^{\text{in}} \dot{m}^{\text{in}} v^{\text{in}}$ on system

Combining modifications #1 and #2

- combined rate of energy flow due to mass inflow $\dot{m}^{\text{in}}$ is

$$\dot{m}^{\text{in}} e^{\text{in}} + p^{\text{in}} \dot{m}^{\text{in}} v^{\text{in}} = \dot{m}^{\text{in}} \left[\frac{1}{2} (\dot{r}^{\text{in}})^2 + g z^{\text{in}} + \underbrace{u^{\text{in}} + p^{\text{in}} v^{\text{in}}}_{h^{\text{in}}} \right]$$

$$= \dot{m}^{\text{in}} \left[\frac{1}{2} (\dot{r}^{\text{in}})^2 + g z^{\text{in}} + h^{\text{in}} \right]$$

- this formula is the reason we defined enthalpy, $H = U + pV$

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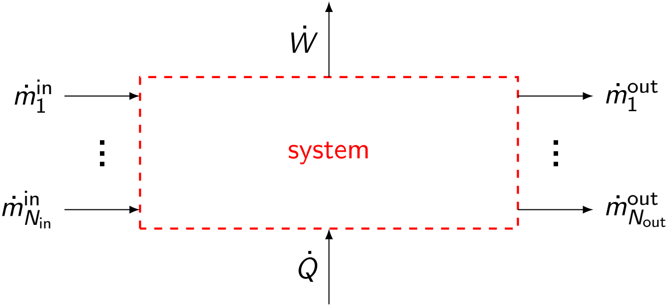
A generic open system



- $\dot{Q}$ is the rate of net heat transfer input (or output if $\dot{Q} < 0$)
- $\dot{W}$ is the rate of net work output (or input if $\dot{W} < 0$)
- $\dot{m}_1^{\text{in}}, \ldots, \dot{m}_{N_{\text{in}}}^{\text{in}} > 0$ are rates of mass input
- $\dot{m}_1^{\text{out}}, \ldots, \dot{m}_{N_{\text{out}}}^{\text{out}} > 0$ are rates of mass output

Rate form

- putting it all together, rate form of 1st law for open systems is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{Q} - \dot{W} + \sum_{i=1}^{N^{\text{in}}} \dot{m}_i^{\text{in}} \left[\frac{1}{2}(\dot{r}_i^{\text{in}})^2 + g z_i^{\text{in}} + h_i^{\text{in}}\right]$$

$$- \sum_{j=1}^{N^{\text{out}}} \dot{m}_j^{\text{out}} \left[\frac{1}{2}(\dot{r}_j^{\text{out}})^2 + g z_j^{\text{out}} + h_j^{\text{out}}\right]$$

Steady-state form

- in steady state, $\mathrm{d}E/\mathrm{d}t = 0$, so rate form of 1st law becomes

$$\dot{Q} + \sum_{i=1}^{N^{\text{in}}} \dot{m}_i^{\text{in}} \left[\frac{1}{2} (\dot{r}_i^{\text{in}})^2 + g z_i^{\text{in}} + h_i^{\text{in}} \right]$$
$$= \dot{W} + \sum_{j=1}^{N^{\text{out}}} \dot{m}_j^{\text{out}} \left[\frac{1}{2} (\dot{r}_j^{\text{out}})^2 + g z_j^{\text{out}} + h_j^{\text{out}} \right]$$

Steady-state, single-input, single-output form

- recall that in steady state, $\mathrm{d}m/\mathrm{d}t = 0$, so

$$\sum_{i=1}^{N_{\text{in}}} \dot{m}_i^{\text{in}} = \sum_{j=1}^{N_{\text{out}}} \dot{m}_j^{\text{out}}$$

- if $N^{\text{in}} = N^{\text{out}} = 1$, then in steady state, $\dot{m}_1^{\text{in}} = \dot{m}_1^{\text{out}} = \dot{m}$
- so steady-state form of 1st law simplifies to

$$\dot{m}\left[\frac{1}{2}(\dot{r}_{\text{in}}^2 - \dot{r}_{\text{out}}^2) + g(z_{\text{in}} - z_{\text{out}}) + h_{\text{in}} - h_{\text{out}}\right] = \dot{W} - \dot{Q}$$

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Problem statement

Upstream of a wind turbine with cross-sectional area A, the air velocity is $\dot{r}_{\text{in}}$. Assuming heat transfer and changes in the air's enthalpy and potential energy are negligible, what is the turbine's maximum mechanical power output? Use the fact that the air velocity at the turbine is the average of the upstream and downstream velocities.

Given and find

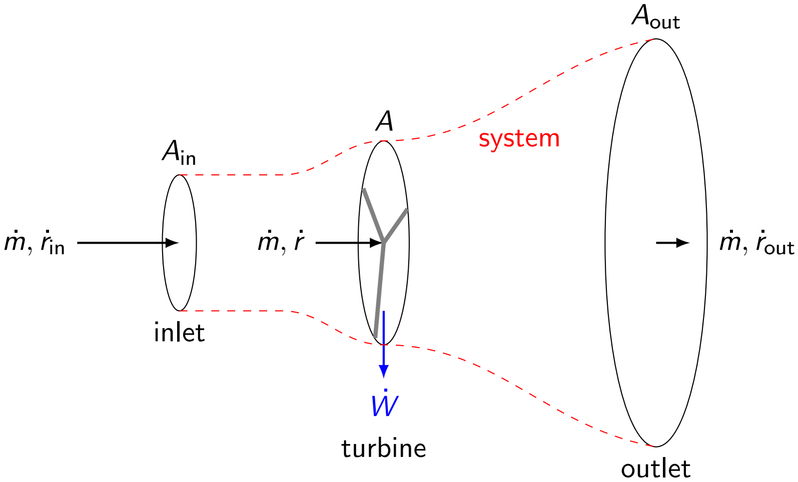
- **given:**
  - ⋄ upstream inlet velocity $\dot{r}_{\text{in}}$
  - ⋄ turbine cross-sectional area A
- **find:**
  - ⋄ maximum power output, $\dot{W}^\star$

Assumptions and basic equations

- **assume:**
  - ◇ steady state
  - ◇ one-dimensional flow at inlet, turbine, outlet
  - ◇ no change in potential energy ($z_{in} = z_{out}$)
  - ◇ no change in specific enthalpy ($h_{in} = h_{out}$)
  - ◇ no heat transfer ($\dot{Q} = 0$)
  - ◇ $\dot{r} = (\dot{r}_{in} + \dot{r}_{out})/2$
- **basic equations:**
  - ◇ for one-dimensional flow, $\dot{m} = \rho A \dot{r}$
  - ◇ 1st law in steady state with 1 mass input and 1 mass output:

$$\dot{m}\left[\frac{1}{2}(\dot{r}_{in}^2 - \dot{r}_{out}^2) + g(z_{in} - z_{out}) + h_{in} - h_{out}\right] = \dot{W} - \dot{Q}$$

System diagram



Solution

- from the 1st law, mechanical power output is

$$\dot{W} = \frac{1}{2}\dot{m}\left(\dot{r}_{\text{in}}^2 - \dot{r}_{\text{out}}^2\right)$$

- from the one-dimensional flow assumptions, mass flow is

$$\dot{m} = \rho_{\text{in}} A_{\text{in}} \dot{r}_{\text{in}} = \rho A \dot{r} = \rho_{\text{out}} A_{\text{out}} \dot{r}_{\text{out}}$$

- but $\dot{r} = (\dot{r}_{\text{in}} + \dot{r}_{\text{out}})/2$, so $\dot{m} = \rho A(\dot{r}_{\text{in}} + \dot{r}_{\text{out}})/2$ and

$$\dot{W} = \frac{1}{4}\rho A(\dot{r}_{\text{in}} + \dot{r}_{\text{out}})\left(\dot{r}_{\text{in}}^2 - \dot{r}_{\text{out}}^2\right)$$

Solution (continued)

- now for some algebra:

$$\begin{aligned}\dot{W} &= \frac{1}{4}\rho A(\dot{r}_{\text{in}} + \dot{r}_{\text{out}})\left(\dot{r}_{\text{in}}^2 - \dot{r}_{\text{out}}^2\right) \\ &= \frac{1}{4}\rho A\left(\dot{r}_{\text{in}}^3 - \dot{r}_{\text{in}}\dot{r}_{\text{out}}^2 + \dot{r}_{\text{in}}^2\dot{r}_{\text{out}} - \dot{r}_{\text{out}}^3\right) \\ &= \frac{1}{4}\rho A\dot{r}_{\text{in}}^3\left(1 - \frac{\dot{r}_{\text{out}}^2}{\dot{r}_{\text{in}}^2} + \frac{\dot{r}_{\text{out}}}{\dot{r}_{\text{in}}} - \frac{\dot{r}_{\text{out}}^3}{\dot{r}_{\text{in}}^3}\right)\end{aligned}$$

- defining $\gamma = \dot{r}_{\text{out}}/\dot{r}_{\text{in}}$, the term in parentheses is

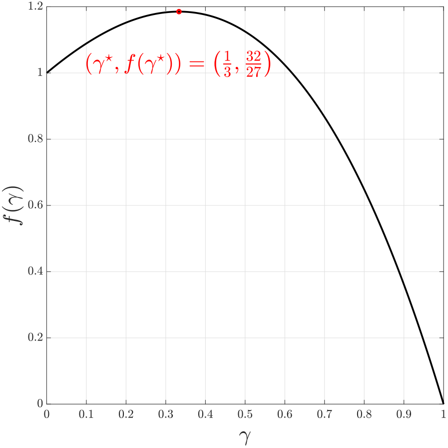
$$f(\gamma) = 1 - \gamma^2 + \gamma - \gamma^3,$$

and the mechanical power output is

$$\dot{W} = \frac{1}{4}\rho A\dot{r}_{\text{in}}^3 f(\gamma)$$

Solution (continued)

- to maximize power, maximize f over γ:



- to calculate $\gamma^\star$, set $f'(\gamma^\star) = 0$ and solve

Solution (continued)

- after maximizing f over γ, maximum power output is

$$\dot{W}^\star = \frac{1}{4}\rho A \dot{r}_{\text{in}}^3 f(\gamma^\star) = \frac{1}{4}\rho A \dot{r}_{\text{in}}^3 \left(\frac{32}{27}\right) = \frac{16}{27}\left(\frac{1}{2}\rho A \dot{r}_{\text{in}}^3\right)$$

- a wind turbine's **coefficient of performance** is defined as

$$C = \frac{\dot{W}}{\frac{1}{2}\rho A \dot{r}_{\text{in}}^3}$$

- any real wind turbine has $\dot{W} \le \dot{W}^\star$, so

$$C = \frac{\dot{W}}{\frac{1}{2}\rho A \dot{r}_{\text{in}}^3} \le \frac{\dot{W}^\star}{\frac{1}{2}\rho A \dot{r}_{\text{in}}^3} = \frac{16}{27} = 59.3\%$$

- this is called the **Betz limit**

Quick follow-up question

Q: As of 2021, the world's biggest wind turbine had 111 m blades. If the upstream air speed is 10.5 m/s and the turbine's coefficient of performance is 50%, what is its mechanical power output?

A: The mechanical power output is

$$\dot{W} = \frac{1}{2} C \rho A \dot{r}_{\text{in}}^3 = \cdots = 14.5\text{MW}$$

(enough to power 2–3000 homes)