Lecture 13 – Conservation of mass

Purdue ME 200, Thermodynamics I

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Outline

Open and closed systems systems

- a **closed system** or **control mass** is a fixed quantity of matter
- an **open system** or **control volume** is a region of space
- the boundary of a control volume is called a **control surface**

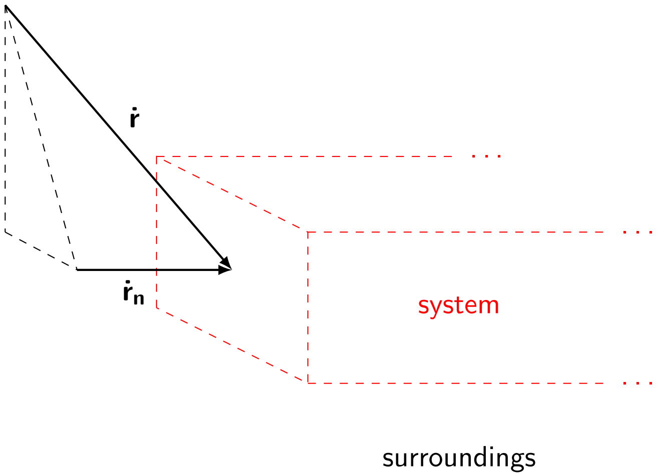
	open	closed
energy crosses boundary?	yes	yes
matter crosses boundary?	yes	no

Velocity and volume notation

- Moran et al. use V for velocity in this chapter
- but we already use V for volume and v for specific volume
- so we'll call velocity $\dot{\mathbf{r}}$
- this follows mechanics notation:
  - ◇ position vector is $\mathbf{r}$
  - ◇ velocity vector is $\dot{\mathbf{r}}$
  - ◇ magnitude of velocity vector is $\dot{r} = \|\dot{\mathbf{r}}\| = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}$

Normal component of velocity

- the subscript n means **normal** to system boundary
- specifically, $\dot{r}_n$ is
  - the magnitude
  - of the component of the velocity vector
  - along the normal vector to the boundary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ṙn
system
surroundings

Outline

Background and notation

Calculating mass flows

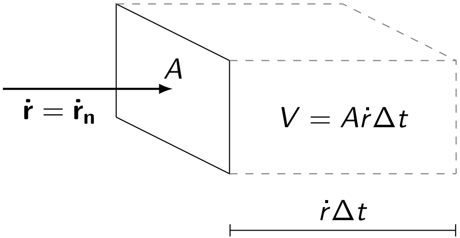
Conservation of mass

Example

One-dimensional flow

- in the one-dimensional flow model,
  - ◇ mass flows are always normal to boundary (so $\dot{\mathbf{r}}_{\mathbf{n}} = \dot{\mathbf{r}}$)
  - ◇ intensive properties are uniform over entrance/exit surfaces
  - $\nearrow$ this includes density and velocity of mass flow $\nwarrow$

Calculating one-dimensional mass flows



- consider one-dimensional flow across a surface of area A
- in time Δt, mass $\Delta m = \rho A\dot{r}\Delta t$ crosses surface
- as $\Delta t \to 0$,

$$\frac{\Delta m}{\Delta t} \to \dot{m} = \rho A\dot{r}$$

⋆ don't mix up density ρ (kg/m^3) and pressure p (kPa)

Calculating non-one-dimensional mass flows

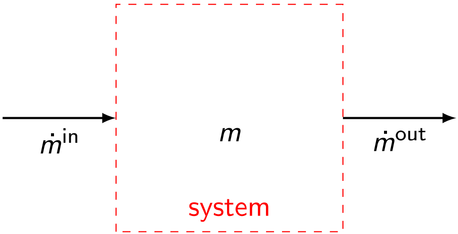
- if the one-dimensional assumptions don't hold, then

$$\dot{m} = \iint_B \rho \dot{r}_n \mathrm{d}A$$

- B is the system boundary (in general, a 2D surface in 3D)
- $\mathrm{d}A$ is a differential surface area element
- ρ and $\dot{r}_n$ may vary along the boundary
- you'll rarely if ever need this form; if you do, see the textbook

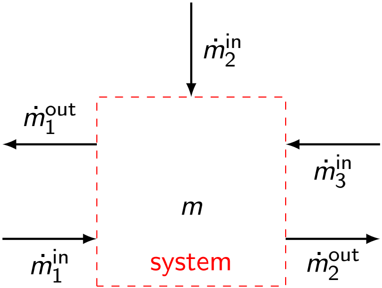
Outline

Single-input, single-output form



$$\frac{\mathrm{d}m}{\mathrm{d}t} = \dot{m}^{\text{in}} - \dot{m}^{\text{out}}$$

Multiple-input, multiple-output form



$$\frac{\mathrm{d}m}{\mathrm{d}t} = \sum_{i=1}^{N_{\text{in}}} \dot{m}_i^{\text{in}} - \sum_{j=1}^{N_{\text{out}}} \dot{m}_j^{\text{out}}$$

Steady-state form

- in steady state, all properties are constant with respect to time
- in particular, mass (a property) is constant, so $\mathrm{d}m/\mathrm{d}t = 0$ and

$$\sum_{i=1}^{N_{\text{in}}} \dot{m}_i^{\text{in}} = \sum_{j=1}^{N_{\text{out}}} \dot{m}_j^{\text{out}}$$

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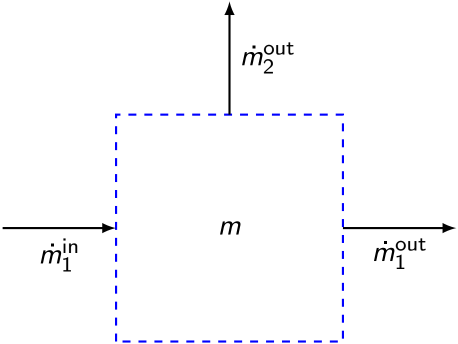
Problem statement

Lake Mead, the reservoir above the Hoover Dam on the Colorado River, initially contains 35 billion m^3 of water. Lake Mead receives 10 billion m^3 per year from the Glen Canyon Dam and other tributaries. The Hoover Dam releases water at an average speed of 0.5 m/s through a cross-sectional area of 700 m^2. Lake Mead loses 500 million m^3 per year via evaporation. How long until Lake Mead runs dry?


UTAH
COLORADO
CALIFORNIA
NEW MEXICO
MEXICO
Grand Junction
COLORADO RIVER
Wayne N. Aspinall Unit
Crystal Dam
Morrow Point Dam
Blue Mesa Dam
GUNNISON RIVER
PRICE RIVER
SAN RAFAEL
FREMONT RIVER
DIRTY DEVIL
ESCALANTE RIVER
DOLORES RIVER
SAN MIGUEL RIVER
SAN JUAN RIVER
ANIMAS RIVER
Navajo Dam
Glen Canyon Dam
Page
Lake Powell
PARIA RIVER
VIRGIN RIVER
MUDDY CREEK
Las Vegas
Lake Mead
Hoover Dam
COLORADO RIVER
LITTLE COLORADO RIVER
Davis Dam
Lake Havasu
Parker Dam
BILL WILLIAMS RIVER
CENTRAL ARIZONA PROJECT
AGUA FRIA RIVER
VERDE RIVER
SALT RIVER
Phoenix
Roosevelt Dam
GILA RIVER
SAN PEDRO RIVER
Tucson
Imperial Dam
SANTA CLARA SLOUGH
GULF OF CALIFORNIA
N

System diagram



Given and find

- **given:**
  - ◇ initial volume: $V(0) = 35$ billion m^3
  - ◇ final volume: $V(t) = 0$ m^3
  - ◇ $\dot{V}_1^{\text{in}} = 10$ billion m^3 per year $= 317.1$ m^3/s
  - ◇ at output 1, $\dot{r} = 0.5$ m/s and $A = 700$ m^2
  - ◇ $\dot{V}_2^{\text{out}} = 500$ million m^3 per year $= 15.9$ m^3/s
- **find:**
  - ◇ time t when water volume reaches $V(t) = 0$ m^3

Assumptions and basic equations

- **assume:**
  - one-dimensional flow at output 1
  - time-invariant flow at all inputs and outputs
  - time-invariant, spatially-uniform water density
- **basic equation:**

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \sum_{i=1}^{N_{\text{in}}} \dot{m}_i^{\text{in}} - \sum_{j=1}^{N_{\text{out}}} \dot{m}_j^{\text{out}}$$

Solution

- from conservation of mass,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \dot{m}_1^{\text{in}} - \dot{m}_1^{\text{out}} - \dot{m}_2^{\text{out}}$$

- but $m = \rho V$ and ρ is time-invariant and spatially uniform, so

$$\rho \frac{\mathrm{d}V}{\mathrm{d}t} = \rho \left(\dot{V}_1^{\text{in}} - \dot{V}_1^{\text{out}} - \dot{V}_2^{\text{out}} \right)$$

  or

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \dot{V}_1^{\text{in}} - \dot{V}_1^{\text{out}} - \dot{V}_2^{\text{out}}$$

Solution (continued)

- flow is one-dimensional, so $\dot{m}_1^{\text{out}} = \rho A \dot{r}$, or equivalently,

$$\dot{V}_1^{\text{out}} = \frac{\dot{m}_1^{\text{out}}}{\rho} = A\dot{r} = (700\text{m}^2)(0.5\text{m/s}) = 350\text{m}^3/\text{s}$$

- so volumetric flow rate is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \dot{V}_1^{\text{in}} - \dot{V}_1^{\text{out}} - \dot{V}_2^{\text{out}} \\ &= 317.1\text{m}^3/\text{s} - 350\text{m}^3/\text{s} - 15.9\text{m}^3/\text{s} \\ &= -48.8\text{m}^3/\text{s}\end{aligned}$$

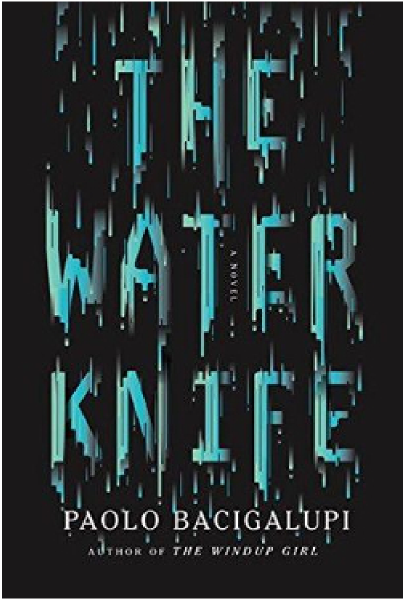
Solution (continued)

- since $\mathrm{d}V/\mathrm{d}t$ is constant,

$$V(t) = V(0) + \left(\frac{\mathrm{d}V}{\mathrm{d}t}\right) t$$

- therefore,

$$t = \frac{V(t) - V(0)}{\mathrm{d}V/\mathrm{d}t} = \frac{0\mathrm{m}^3 - 3.5 \times 10^{10}\mathrm{m}^3}{-48.8\mathrm{m}^3/\mathrm{s}}$$
$$= 22.7 \text{ years}$$


THE
WATER
KNIFE
A NOVEL
PAOLO BACIGALUPI
AUTHOR OF THE WINDUP GIRL