Lecture 4 – Mechanical work and energy

Purdue ME 200, Thermodynamics I

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Outline

Differentials

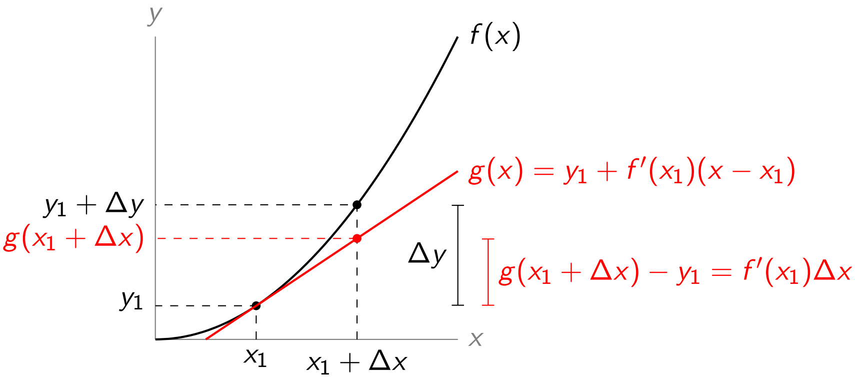
Mechanical work

Conservation of mechanical energy

What are differentials?

- a **differential** $\mathrm{d}x$ is just a tiny (infinitesimal) change in x
- $\mathrm{d}x$ can be viewed as the limit of a finite Δx as $\Delta x \to 0$
- differentials often show up under integral signs
  (to calculate total change, add up a bunch of tiny changes)

Differentials in one dimension



- suppose $y = f(x)$ and g is the linear approximation to f at x_1
- when x changes by Δx, y changes by a true amount Δy
- for small Δx, the true $\Delta y \approx f'(x_1)\Delta x$

Differentials in one dimension (continued)

- as $\Delta x \to 0$, the approximation $\Delta y \approx f'(x_1)\Delta x$ becomes exact
- at any arbitrary x, we write

$$\mathrm{d}y = f'(x)\mathrm{d}x \text{ or } \mathrm{d}y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

- from the fundamental theorem of calculus,

$$\int_{y_1}^{y_2} \mathrm{d}y = \int_{x_1}^{x_2} f'(x)\mathrm{d}x = f(x_2) - f(x_1) = y_2 - y_1$$

- more compactly (but less precisely), we can write

$$\int \mathrm{d}y = \Delta y$$

Differentials in two dimensions

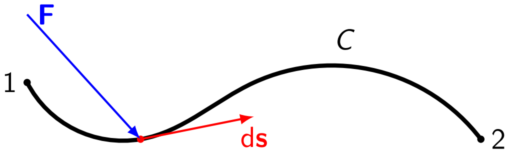
- consider a variable $z = f(x, y)$
- the differential of z is

$$\mathrm{d}z = \left(\frac{\partial f}{\partial x}\right)_y \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_x \mathrm{d}y$$

- in 2D, $\int \mathrm{d}z = \Delta z$ only holds for **exact** differentials
- only properties (p, v, T, $\ldots$) have exact differentials
- other things (work W, heat Q) have **inexact** differentials
- we write inexact differentials with δ instead of d
- we never write $\int \delta W = \Delta W$ or $\int \delta Q = \Delta Q$

Outline

General form of mechanical work



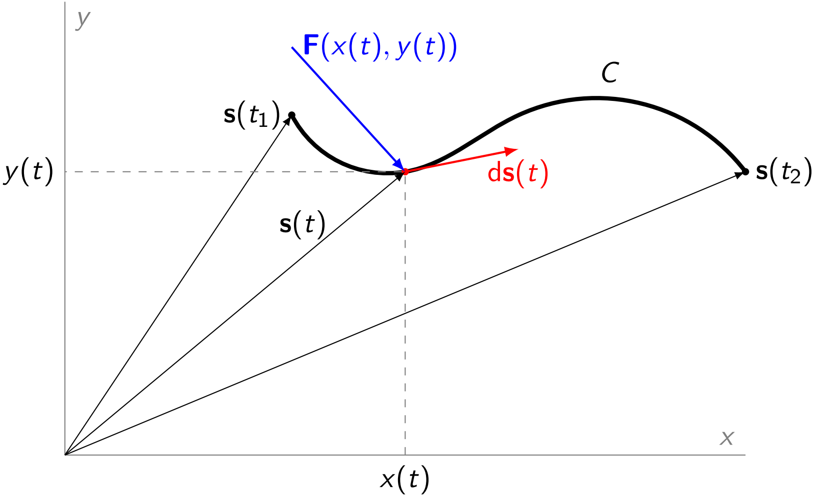
- work done by force **F** on body moved over path C is

$$W = \int_C \mathbf{F} \cdot d\mathbf{s}$$

- differential displacement d**s** always points along path
- dot product picks out the component of **F** along path
- if path starts at point 1 and ends at point 2, we may write

$$W = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$$

Parameterization in terms of t



$$W = \int_{t_1}^{t_2} \mathbf{F}(x(t), y(t)) \cdot d\mathbf{s}(t)$$

Calculating mechanical work

- force can be written as

$$\mathbf{F}(x(t), y(t)) = F_x(x(t), y(t))\mathbf{i} + F_y(x(t), y(t))\mathbf{j}$$

- path C contains all position vectors $\mathbf{s}(t)$ over $t_1 \leq t \leq t_2$
- position vector can be written as $\mathbf{s}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$
- so differential displacement along path is

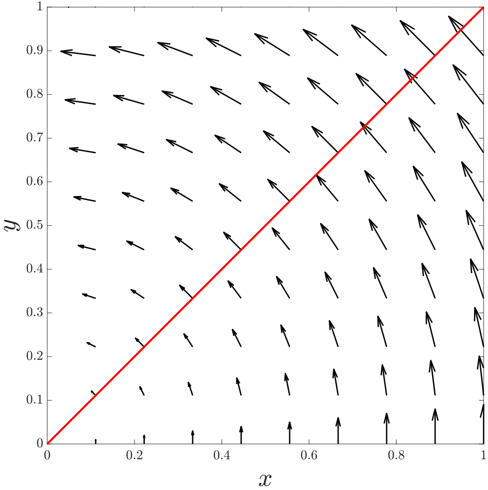
$$\mathrm{d}\mathbf{s}(t) = \mathrm{d}x(t)\mathbf{i} + \mathrm{d}y(t)\mathbf{j} = x'(t)\mathrm{d}t\mathbf{i} + y'(t)\mathrm{d}t\mathbf{j}$$

- and work done by force $\mathbf{F}$ over path C is

$$W = \int_{t_1}^{t_2} \left[F_x(x(t), y(t))x'(t) + F_y(x(t), y(t))y'(t)\right] \mathrm{d}t$$

Example

- suppose force is $\mathbf{F}(x(t), y(t)) = -y(t)\mathbf{i} + x(t)\mathbf{j}$
- and path is $\mathbf{s}(t) = t\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$



Solution

- force is perpendicular to path, so $\mathbf{F} \cdot d\mathbf{s} = 0$ and $W = 0$
- does the math agree?

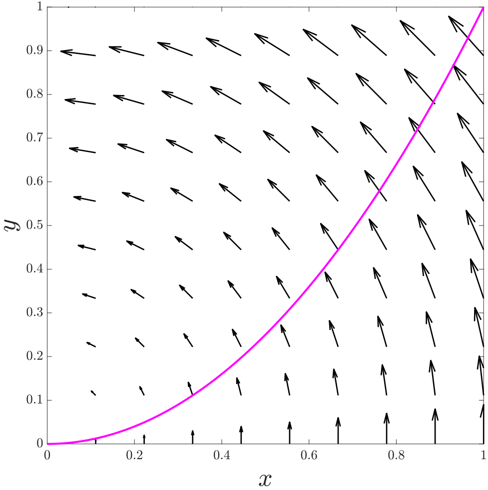
$$
\begin{aligned}
&x(t) = t,\; y(t) = t \\
&x'(t) = 1,\; y'(t) = 1 \\
&F_x(x(t), y(t)) = -y(t) = -t \\
&F_y(x(t), y(t)) = x(t) = t
\end{aligned}
$$

- therefore,

$$
\begin{aligned}
W &= \int_{t_1}^{t_2} \left[F_x(x(t), y(t))x'(t) + F_y(x(t), y(t))y'(t)\right] \mathrm{d}t \\
&= \int_0^1 \left[(-t)(1) + (t)(1)\right] \mathrm{d}t = 0
\end{aligned}
$$

Modified example

- what if path instead is $\mathbf{s}(t) = t\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$



Modified solution

- force is no longer perpendicular to path, so $\mathbf{F} \cdot d\mathbf{s} \neq 0$
- along the modified path,

$$
\begin{aligned}
&x(t) = t,\ y(t) = t^2 \\
&x'(t) = 1,\ y'(t) = 2t \\
&F_x(x(t), y(t)) = -y(t) = -t^2 \\
&F_y(x(t), y(t)) = x(t) = t
\end{aligned}
$$

- so the work is

$$
\begin{aligned}
W &= \int_{t_1}^{t_2} \left[F_x(x(t), y(t))x'(t) + F_y(x(t), y(t))y'(t)\right] dt \\
&= \int_0^1 \left[(-t^2)(1) + (t)(2t)\right] dt \\
&= \int_0^1 \left(2t^2 - t^2\right) dt = \frac{1}{3} \neq 0
\end{aligned}
$$

Work is path-dependent

- these examples show that work is **path-dependent**
- we can't know the work done without knowing the path taken
- this implies that **work is not a property**
- as a reminder, we write differential work as δW, not $\mathrm{d}W$
- and we never write $\int \delta W = \Delta W$

Work and power

- power $\dot{W}$ is defined as the time derivative of work
- from the fundamental theorem of calculus, $W = \int_{t_1}^{t_2} \dot{W} \mathrm{d}t$
- earlier, we found that

$$W = \int_{t_1}^{t_2} \left[F_x(x(t), y(t))x'(t) + F_y(x(t), y(t))y'(t) \right] \mathrm{d}t$$

- the integrand is the dot product of $\mathbf{F}(x(t), y(t))$ and velocity,

$$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

- so $W = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \mathrm{d}t$, and therefore

$$\dot{W} = \mathbf{F} \cdot \mathbf{v}$$

Outline

Differentials

Mechanical work

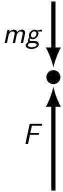
Conservation of mechanical energy

Mechanical work and energy in one dimension

- consider a body of mass m at height y and (vertical) speed v
- its kinetic energy (KE) is $\frac{1}{2}mv^2$
- its gravitational potential energy (PE) is mgy
  (defining PE $= 0$ at $y = 0$)
- suppose some other forces, summing to F, act on the body
- if the body moves from y_1 to y_2, those forces do work

$$W = \int_{y_1}^{y_2} F \mathrm{d}y$$

Conservation of energy in one dimension



- Newton's second law: $F - mg = ma$
- integrate both sides over y from y_1 to y_2:

$$\int_{y_1}^{y_2} (F - mg)\mathrm{d}y = m \int_{y_1}^{y_2} a\mathrm{d}y$$

- the left-hand side is

$$\int_{y_1}^{y_2} (F - mg)\mathrm{d}y = \int_{y_1}^{y_2} F\mathrm{d}y - mg(y_2 - y_1) = W - \Delta \mathrm{PE}$$

Conservation of energy in one dimension (continued)

- the right-hand side is

$$m\int_{y_1}^{y_2} a\mathrm{d}y = m\int_{y_1}^{y_2} \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}y = m\int_{y_1}^{y_2} \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}y = m\int_{y_1}^{y_2} v\frac{\mathrm{d}v}{\mathrm{d}y}\mathrm{d}y$$

- change variables of integration from y to v:
  - ◇ $v = v_1$ at $y = y_1$, and $v = v_2$ at $y = y_2$
  - ◇ since v is a function of y, its differential is $\mathrm{d}v = (\mathrm{d}v/\mathrm{d}y)\mathrm{d}y$
- then the integral becomes

$$m\int_{v_1}^{v_2} v\mathrm{d}v = m\left[\frac{1}{2}v^2\right]_{v_1}^{v_2} = \frac{1}{2}m(v_2^2 - v_1^2) = \Delta\mathrm{KE}$$

Conservation of energy in one dimension (continued)

- putting the left- and right-hand sides together gives

$$\Delta\text{KE} + \Delta\text{PE} = W$$

- this is conservation of (mechanical) energy in one dimension
- although the derivation was 1D, the result holds in 2D and 3D
- this is a special case of the **1st law** of thermodynamics
- we'll progressively generalize the 1st law to include
  - ◇ other types of work
  - ◇ internal energy
  - ◇ heat transfer
  - ◇ mass transfer

Sign convention

- in this derivation, W was work done by forces on body
- we usually take W to be work done by system on surroundings (historically, thermo focused on work done by heat engines)
- with this sign convention, 1st law becomes

$$\Delta \mathrm{KE} + \Delta \mathrm{PE} = -W$$

Example

1,000 kg of water, initially at rest, flows through a vertical displacement of 60 m in a frictionless pipe. At the bottom, the water flows through a turbine, which it exits at 20 m/s.

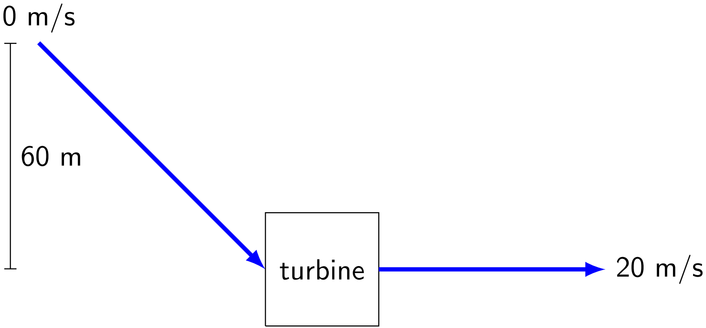
(a) How much mechanical work does the water do on the turbine?

(b) If this takes 10 s, what is the average mechanical power?

Given and find

- **given**
  - ◇ $m = 1000$ kg
  - ◇ $\Delta z = 60$ m
  - ◇ $v_1 = 0$ m/s
  - ◇ $v_2 = 20$ m/s
- **find**
  - (a) mechanical work W done by water on turbine
  - (b) average mechanical power $\dot{W}_{\mathrm{av}}$ over $\Delta t = 10$ s

System diagram



Assumptions and basic equations

- **assumptions**
  - ◇ frictionless pipe (so no energy lost inside)
- **basic equations**
  - ◇ $\Delta\mathrm{KE} + \Delta\mathrm{PE} = -W$
  - ◇ also definitions: $\mathrm{KE} = \frac{1}{2}mv^2$, $\mathrm{PE} = mgz$

Solution

- change in kinetic energy is

$$\Delta \mathrm{KE} = \frac{1}{2} m(v_2^2 - v_1^2) = \frac{1}{2}(1000\mathrm{kg})[(20\mathrm{m/s})^2 - (0\mathrm{m/s})^2] = 200\mathrm{kJ}$$

- change in potential energy is

$$\Delta \mathrm{PE} = mg\Delta z = (1000\mathrm{kg})(10\mathrm{m/s}^2)(0\mathrm{m} - 60\mathrm{m}) = -600\mathrm{kJ}$$

- mechanical work done by water on turbine is

$$W = -(\Delta \mathrm{KE} + \Delta \mathrm{PE}) = 400\mathrm{kJ}$$

- average mechanical power is

$$\dot{W}_{\mathrm{av}} = \frac{W}{\Delta t} = \frac{400\mathrm{kJ}}{10\mathrm{s}} = 40\mathrm{kW}$$