Lecture 11 – Modeling gases

Purdue ME 200, Thermodynamics I

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Outline

What is an ideal gas?

- a collection of point-mass particles
- that do not interact with each other
- and obey classical (not quantum) mechanics

When can we use the ideal gas model?

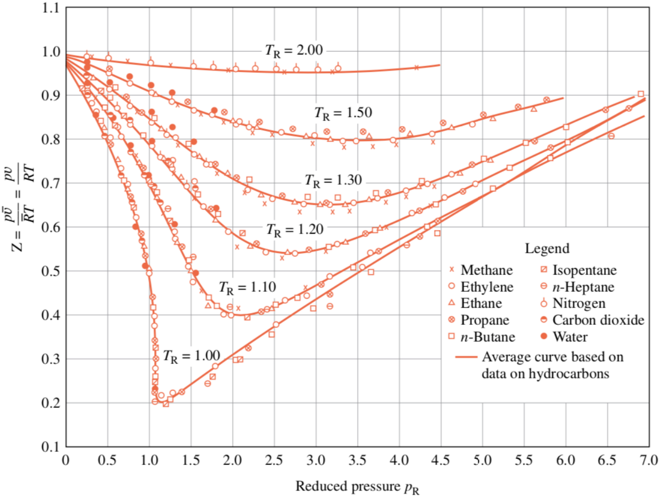
- with most common gases (like air, O_2, N_2, CO_2, CO, H_2)
- when pressure is low ($p \ll p_c$)
- and/or temperature is high ($T \gg T_c$)

Compressibility factor

- the **compressibility factor** is

$$Z = \frac{pv}{RT} = \frac{p\bar{v}}{\bar{R}T}$$

- if $Z = 1$, then $pv = RT$
- so a gas is ∼ideal if $Z \approx 1$



- $p_R = p/p_c$, $T_R = T/T_c$ are reduced pressure, temperature
- $Z \geq \sim 0.9$ (gas is ∼ideal) if $p_R \leq \sim 0.25$ and/or $T_R \geq \sim 1.75$

Moran et al., *Fundamentals of Engineering Thermodynamics* (2018)

Example

- the critical point of air is $T_c \approx 133$ K and $p_c \approx 38$ bar
- so $p_R \leq 0.25$ whenever $p \leq 0.25 p_c \approx 10$ bar
- and $T_R \geq 1.75$ whenever $T \geq 1.75 T_c \approx 233$ K $= -40$ °C

$\implies$ air is $\sim$ideal below 10 bar and/or above -40 °C

Outline

The ideal gas model

- the basic assumptions underlying the ideal gas model are
  - ◇ the equation of state, $pv = RT$
  - ◇ internal energy depends only on temperature, $u = u(T)$
- it follows that enthalpy depends only on temperature:

$$h = u + pv = u(T) + RT = h(T)$$

⋆ equation of state can take various forms via the definitions

$$v = \frac{V}{m},\ \bar{v} = \frac{V}{n},\ M = \frac{m}{n},\ \bar{R} = MR$$

Ideal gas specific heats

- specific heats depend only on temperature:

$$c_v(T,v) = \left(\frac{\partial}{\partial T} u(T)\right)_v = \frac{\mathrm{d}u}{\mathrm{d}T} = c_v(T)$$
$$c_p(T,p) = \left(\frac{\partial}{\partial T} h(T)\right)_p = \frac{\mathrm{d}h}{\mathrm{d}T} = c_p(T)$$

- differentiating both sides of $h(T) = u(T) + RT$ gives

$$\frac{\mathrm{d}h}{\mathrm{d}T} = \frac{\mathrm{d}u}{\mathrm{d}T} + R$$

- so given one specific heat, we can always find the other from

$$c_p(T) = c_v(T) + R$$

Ideal gas specific heat ratio

- in terms of specific heat ratio $k(T) = c_p(T)/c_v(T)$,

$$\begin{aligned} c_p(T) &= c_v(T) + R = \frac{c_p(T)}{k(T)} + R \\ \iff c_p(T)\left(1 - \frac{1}{k(T)}\right) &= R \\ \iff c_p(T) &= \frac{R}{1 - 1/k(T)} \end{aligned}$$

- similarly,

$$c_v(T) = \frac{R}{k(T) - 1}$$

Ideal gas internal energy and enthalpy changes

- from the fundamental theorem of calculus,

$$\Delta u = u(T_2) - u(T_1) = \int_{T_1}^{T_2} \frac{\mathrm{d}u}{\mathrm{d}T}\mathrm{d}T = \int_{T_1}^{T_2} c_v(T)\mathrm{d}T$$

$$\Delta h = h(T_2) - h(T_1) = \int_{T_1}^{T_2} \frac{\mathrm{d}h}{\mathrm{d}T}\mathrm{d}T = \int_{T_1}^{T_2} c_p(T)\mathrm{d}T$$

- if $T_2 - T_1$ is not too large, then for all T between T_1 and T_2,
  - ◇ $T \approx T_{\mathrm{av}} = (T_1 + T_2)/2$
  - ◇ $c_v(T) \approx c_v(T_{\mathrm{av}})$ and $c_p(T) \approx c_p(T_{\mathrm{av}})$
- in this case, $\Delta u \approx c_v(T_{\mathrm{av}})\Delta T$ and $\Delta h \approx c_p(T_{\mathrm{av}})\Delta T$

Outline

Problem statement

The air in a car tire, initially at 20 °C and 221 kPa (gage), warms to 40 °C after the car drives for a while. What is the new pressure of the air inside the tire?

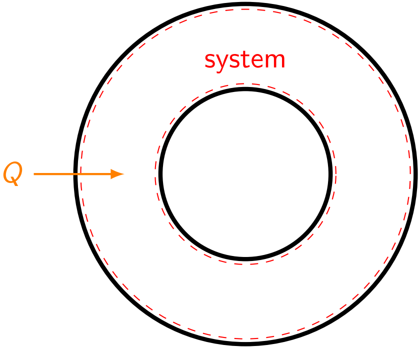
Given and find

- **given:**
  - $T_1 = 20\ ^\circ\text{C}$, $p_1 = 221$ kPa (gage)
  - $T_2 = 40\ ^\circ\text{C}$
- **find:**
  - p_2

Assumptions and basic equations

- **assume:**
  - closed system
  - constant volume ($v_1 = v_2$)
  - ideal gas ($p_1 v_1 = RT_1$, $p_2 v_2 = RT_2$)
- **basic equations:** none

System diagram



Solution

- from the ideal gas law, $p_1 v_1 = RT_1$ and $p_2 v_2 = RT_2$
- but $v_1 = v_2$ and R is constant, so

$$\frac{v_1}{R} = \frac{T_1}{p_1} = \frac{T_2}{p_2} = \frac{v_2}{R}$$

- rearranging the middle equation,

$$p_2 = \frac{p_1 T_2}{T_1} = \frac{(221\text{kPa})(40^\circ\text{C})}{20^\circ\text{C}} = 442\text{kPa ????}$$

- nope! always use **absolute** T and p in the ideal gas law

$$p_2 = \frac{p_1 T_2}{T_1} = \frac{(322\text{kPa})(331\text{K})}{311\text{K}} = 343\text{kPa}$$

Outline

Ideal gases

The ideal gas model

Example #1

Example #2

Problem statement

Air, initially occupying 700 ft^3 at 120 $^\circ$F, cools to 70 $^\circ$F in one minute at constant, atmospheric pressure.

(a) What is the average rate of heat transfer?

(b) What is the average power associated with boundary work?

(c) How much does the air's internal energy change?

Given and find

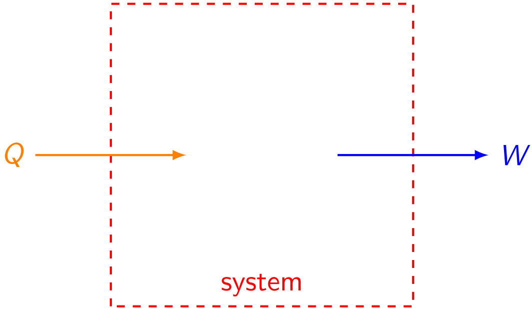
- **given:**
  - ◇ $T_1 = 120\ ^\circ\text{F} = 322\ \text{K},\ V_1 = 700\ \text{ft}^3 = 19.8\ \text{m}^3$
  - ◇ $T_2 = 70\ ^\circ\text{F} = 294\ \text{K}$
  - ◇ $p = p_{\text{atm}} = 101\ \text{kPa}$ (constant)
  - ◇ $\Delta t = 1\ \text{min} = 60\ \text{s}$
- **find:**
  - (a) $Q/\Delta t$
  - (b) $W/\Delta t$
  - (c) ΔU

Assumptions and basic equations

- **assume:**
  - ◇ closed system (no mass transfer)
  - ◇ stationary ($\Delta\text{KE} = \Delta\text{PE} = 0$)
  - ◇ ideal gas ($pv = RT$)
  - ◇ constant pressure ($W = p\Delta V$ and $\Delta H = \Delta U + p\Delta V$)
  - ◇ constant specific heats ($\Delta u = c_v(T_{\text{av}})\Delta T$, $\Delta h = c_p(T_{\text{av}})\Delta T$)
- **basic equation:**

$$\Delta\text{KE} + \Delta\text{PE} + \Delta U = Q - W$$

System diagram



Solution to part (a)

- from the 1st law with $\Delta\text{KE} = \Delta\text{PE} = 0$ and $W = p\Delta V$,

$$Q = \Delta U + W = \Delta U + p\Delta V = \Delta H = m\Delta h$$

- but $m = V_1/v_1$ and $\Delta h = c_p(T_{\text{av}})\Delta T$, so

$$Q = \frac{V_1 c_p(T_{\text{av}})\Delta T}{v_1}$$

- from the ideal gas law, $v_1 = RT_1/p$, so

$$Q = \frac{V_1 p c_p(T_{\text{av}})\Delta T}{RT_1}$$

Solution to part (a) (continued)

- for air at $T_{\text{av}} = 95\ ^\circ\text{F}$, $c_p(T_{\text{av}}) = 1.01\ \text{kJ}/(\text{kg K})$
- gas constant for air is $R = 0.287\ \text{kJ}/(\text{kg K})$
- so average rate of heat transfer is

$$
\begin{aligned}
\frac{Q}{\Delta t} &= \frac{V_1 p c_p(T_{\text{av}})(T_2 - T_1)}{R T_1 \Delta t} \\
&= \frac{(19.8\text{m}^3)(101\text{kPa})(1.01\text{kJ}/(\text{kg K}))(294\text{K} - 322\text{K})}{(0.287\text{kJ}/(\text{kg K}))(322\text{K})(60\text{s})} \\
&= -10.2\text{kW}
\end{aligned}
$$

- Q is negative, so heat transfers from air to surroundings

Solution to part (b)

- average power is $W/\Delta t = p(V_2 - V_1)/\Delta t$
- since pressure is constant,

$$p = \frac{RT_1}{v_1} = \frac{RT_2}{v_2} \implies v_2 = \frac{v_1 T_2}{T_1}$$

- but mass is constant, so $v_1 = V_1/m$, $v_2 = V_2/m$ and

$$V_2 = mv_2 = \frac{mv_1 T_2}{T_1} = \frac{m(V_1/m)T_2}{T_1} = \frac{V_1 T_2}{T_1}$$

- so average power is

$$\frac{W}{\Delta t} = \frac{p(V_2 - V_1)}{\Delta t} = \frac{p[(V_1 T_2/T_1) - V_1]}{\Delta t} = \frac{pV_1(T_2/T_1 - 1)}{\Delta t}$$
$$= \frac{(101\text{kPa})(19.8\text{m}^3)[(294\text{K})/(322\text{K}) - 1]}{60\text{s}} = -2.9\text{kW}$$

Solution to part (c)

- from the 1st law,

$$\Delta U = Q - W = \Delta t(Q/\Delta t - W/\Delta t)$$
$$= (60\text{s})[-10.2\text{kW} - (-2.9\text{kW})] = -438\text{kJ}$$

- can check this via $\Delta u = c_v(T_{\text{av}})\Delta T$:

$$\Delta U = mc_v(T_{\text{av}})(T_2 - T_1) = \frac{V_1 p c_v(T_{\text{av}})(T_2 - T_1)}{RT_1}$$
$$= \cdots = -435\text{kJ}$$

What did we learn?

- 700 ft^3 of air at 120 $^\circ$F and p_{atm} cooled to 70 $^\circ$F in 1 min
- rate of heat transfer was

$$\frac{Q}{\Delta t} = \frac{mc_p(T_{\text{av}})\Delta T}{\Delta t} \approx -10\text{kW or } -34{,}000 \text{ BTU/h}$$

⋆ the related formula

$$\dot{Q} \approx \dot{m}c_p\Delta T$$

is widely used in thermal engineering