

# Lecture 22 – The Carnot cycle

Purdue ME 200, Thermodynamics I

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# Outline

The Carnot cycle

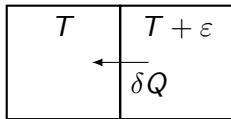
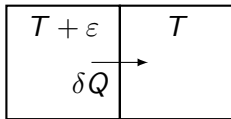
Example

# Motivation

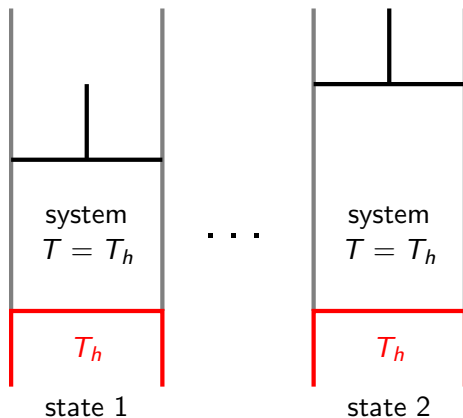
- many ideas around the 2nd law involve reversible cycles
- a reversible cycle must
  - ◇ have no internal irreversibilities (friction, etc.)
  - ◇ have only reversible heat transfer with the surroundings
- can we create a reversible cycle?

## Reminder: (approximately) reversible heat transfer

- real (irreversible) heat transfer is always driven by a finite  $\Delta T$
- but heat transfer approaches reversibility as  $\Delta T \rightarrow 0$
- systems at  $T_a$  and  $T_b$  exchange heat  $\sim$ reversibly if  $T_a \approx T_b$

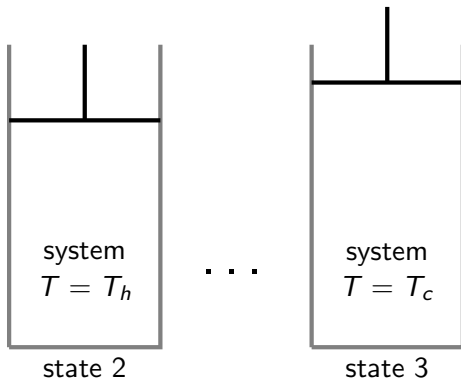


## Carnot power cycle process 1-2: Isothermal expansion



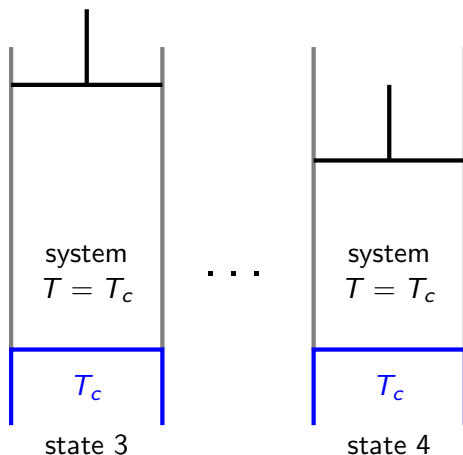
- constant system temperature  $T = T_h$
- reversible heat transfer  $Q_h$  from hot reservoir to system
- no other heat transfer (cylinder walls are well-insulated)

## Carnot power cycle process 2-3: Adiabatic expansion



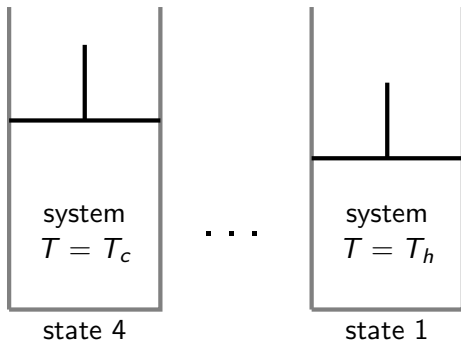
- no heat transfer
- system temperature drops from  $T_h$  to  $T_c$

## Carnot power cycle process 3-4: Isothermal compression



- constant system temperature  $T = T_c$
- reversible heat transfer  $Q_c$  from system to hot reservoir
- no other heat transfer

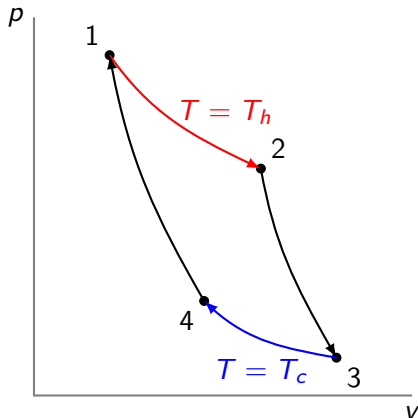
## Carnot power cycle process 4-1: Adiabatic compression



- no heat transfer
- system temperature rises from  $T_c$  to  $T_h$



## Carnot power cycle $p$ - $v$ diagram



- the area under each curve is the work done over that process
- the total work done is the area enclosed by the cycle

# Carnot cycle notes

- this piston-cylinder Carnot cycle is one implementation
- others are possible, such as a phase-change cycle using
  - ◇ 1-2: isothermal evaporation in a boiler
  - ◇ 2-3: adiabatic expansion in a turbine
  - ◇ 3-4: isothermal condensation in a condenser
  - ◇ 4-1: adiabatic compression in a pump
- all Carnot cycle implementations
  - ◇ use two isothermal processes
  - ◇ use two adiabatic processes
  - ◇ have no internal irreversibilities
- backwards Carnot power cycle = Carnot refrigeration cycle

# Outline

The Carnot cycle

Example

## Problem statement

Calculate the efficiency of a Carnot power cycle implemented in a piston-cylinder device containing an ideal gas. The cycle runs between reservoirs at temperatures  $T_h$  and  $T_c$ .

## Short solution

- all implementations of the Carnot cycle are reversible
- all reversible power cycles have efficiency  $\eta = 1 - T_c/T_h$

# Long solution: Assumptions and basic equations

- **assume:**

- ◇ ideal gas
- ◇ closed system ( $\Delta m = 0$ )
- ◇ stationary system ( $\Delta KE = \Delta PE = 0$ )
- ◇ processes 1-2 and 3-4 are isothermal at  $T_h$  and  $T_c$
- ◇ processes 2-3 and 4-1 are adiabatic ( $Q_{23} = Q_{41} = 0$ )

- **basic equations:**

- ◇ 1st law for closed stationary systems:  $\Delta U = Q - W$
- ◇ ideal gas law:  $pV = mRT$
- ◇ boundary work:  $W = \int_{V_1}^{V_2} p dV$
- ◇ power cycle efficiency:  $\eta = W/Q_h$

## Long solution

- efficiency is  $\eta = W/Q_h$ , where  $Q_h = Q_{12}$
- net work is

$$W = W_{12} + W_{23} + W_{34} + W_{41}$$

- in isothermal ideal gas expansion 1-2 with  $T = T_h$ ,

$$W_{12} = \int_{V_1}^{V_2} p dV = mRT_h \int_{V_1}^{V_2} \frac{dV}{V} = mRT_h \ln \left( \frac{V_2}{V_1} \right)$$

- 3-4 is similar except  $T = T_c$ , so

$$W_{34} = mRT_c \ln \left( \frac{V_4}{V_3} \right)$$

## Long solution (continued)

- 2-3 is adiabatic ( $Q_{23} = 0$ ), so from 1st law,

$$W_{23} = U_3 - U_2 = m(u_3 - u_2)$$

- but  $u = u(T)$  for an ideal gas,  $T_3 = T_c$ , and  $T_2 = T_h$ , so

$$W_{23} = m(u(T_c) - u(T_h))$$

- similarly, for adiabatic process 4-1,

$$W_{41} = m(u(T_h) - u(T_c)) = -W_{23}$$



## Long solution (continued)

- since  $W_{41} = -W_{23}$ , net work is

$$W = W_{12} + \cancel{W_{23}} + W_{34} + \cancel{W_{41}} = W_{12} + W_{34}$$

- efficiency  $W/Q_h$  also requires input heat transfer  $Q_h = Q_{12}$
- but 1-2 is isothermal and  $u = u(T)$  for ideal gas, so  $U_2 = U_1$
- 1st law therefore gives

$$Q_{12} = W_{12}$$

## Long solution (continued)

- from net work and input heat transfer formulas,

$$\begin{aligned}\eta &= \frac{W}{Q_h} = \frac{W_{12} + W_{34}}{W_{12}} = 1 + \frac{W_{34}}{W_{12}} \\ &= 1 + \frac{mR T_c \ln(V_4/V_3)}{mR T_h \ln(V_2/V_1)}\end{aligned}$$

- this looks almost like our reversible efficiency,  $1 - T_c/T_h$
- it's exactly right if  $\ln(V_4/V_3) = -\ln(V_2/V_1)$
- but  $-\ln(x) = \ln(1/x)$ , so it's exactly right if  $V_4/V_3 = V_1/V_2$

## Long solution (continued)

- fact: adiabatic ideal gas expansion/compression is polytropic
- so for adiabatic processes 2-3 and 4-1 (see lecture 12 slide 5),

$$\left(\frac{V_3}{V_2}\right)^{n-1} = \frac{T_2}{T_3} \text{ and } \left(\frac{V_1}{V_4}\right)^{n-1} = \frac{T_4}{T_1}$$

- but  $T_1 = T_2 = T_h$  and  $T_3 = T_4 = T_c$ , so

$$\frac{T_c}{T_h} = \left(\frac{V_1}{V_4}\right)^{n-1} = \left(\frac{V_2}{V_3}\right)^{n-1}$$

- taking the  $n - 1$  roots and rearranging gives  $V_4/V_3 = V_1/V_2$
- so the ideal gas Carnot power cycle efficiency is

$$\eta = 1 - \frac{T_c}{T_h}$$

## Addendum: Adiabatic ideal gas expansion is polytropic

- for an ideal gas,  $dU = mc_v dT$
- closed stationary adiabatic 1st law:  $dU = \cancel{\delta Q} - \delta W = -pdV$
- so  $mc_v dT = -pdV$ , or equivalently,  $mdT = -(p/c_v)dV$
- for an ideal gas,  $pV = mRT$
- taking total differential of both sides,

$$Vdp + pdV = mRdT$$

## Adiabatic ideal gas expansion is polytropic (continued)

- combining the last equation with  $mdT = -(p/c_v)dV$ ,

$$\begin{aligned}Vdp + pdV &= -(Rp/c_v)dV \\ \iff Vdp + p(1 + R/c_v)dV &= 0\end{aligned}$$

- dividing through by  $pV$ ,

$$\frac{dp}{p} + \frac{(1 + R/c_v)dV}{V} = 0$$

## Adiabatic ideal gas expansion is polytropic (continued)

- the left-hand side is the total differential of

$$f(p, V) = \ln(p) + (1 + R/c_v) \ln(V)$$

- since  $df = 0$ ,  $f$  must equal some constant (call it  $\alpha$ ):

$$\ln(p) + (1 + R/c_v) \ln(V) = \alpha$$

$$\iff \ln(p) + \ln(V^{1+R/c_v}) = \alpha$$

$$\iff \ln(pV^{1+R/c_v}) = \alpha$$

$$\iff pV^{1+R/c_v} = e^\alpha$$

- so the process is polytropic with coefficient  $n = 1 + R/c_v$
- for ideal gases,  $R/c_v = k - 1$  (where  $k = c_p/c_v$ ), so  $n = k$