

# Lecture 26 – Ideal gas entropy changes

Purdue ME 200, Thermodynamics I

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# Outline

Integrating exact differentials

Ideal gas entropy changes

Example

## A problem in the ideal gas entropy change derivation

- we have some variable  $z = f(x, y)$  and want to calculate

$$\Delta z = z_2 - z_1 = f(x_2, y_2) - f(x_1, y_1)$$

- we don't know  $f$ , but we know  $\partial f / \partial x$  and  $\partial f / \partial y$

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- we don't know  $f$ , but we know  $\partial f / \partial x$  and  $\partial f / \partial y$
- in 1D, we could use the fundamental theorem of calculus:

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x) dx$$

- in 2D, it's a little more complicated

## Applying path-independence

- since  $\partial f/\partial x$  and  $\partial f/\partial y$  exist,  $z$  has an exact differential:

$$dz = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

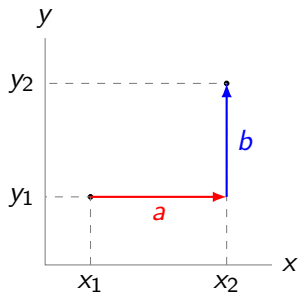
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- any variable with an exact differential is path-independent
- so we can calculate  $\Delta z$  by integrating over any path we like
- a particularly simple path:



- (a) first  $x_1 \rightarrow x_2$  with  $y = y_1$
- (b) then  $y_1 \rightarrow y_2$  with  $x = x_2$

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- the change in  $z$  along (vertical) path **b** is

$$\Delta z_b = f(x_2, y_2) - f(x_2, y_1) = \int_{y_1}^{y_2} \left( \frac{\partial f}{\partial y} \right)_{x=x_2} dy$$



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$$\Delta z_b = f(x_2, y_2) - f(x_2, y_1) = \int_{y_1}^{y_2} \left( \frac{\partial f}{\partial y} \right)_{x=x_2} dy$$

- so the total change in  $z$ ,  $\Delta z = \Delta z_a + \Delta z_b$ , is

$$f(x_2, y_2) - f(x_1, y_1) = \int_{x_1}^{x_2} \left( \frac{\partial f}{\partial x} \right)_{y=y_1} dx + \int_{y_1}^{y_2} \left( \frac{\partial f}{\partial y} \right)_{x=x_2} dy$$

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## Reminder: the ideal gas model

- the basic assumptions underlying the ideal gas model are
  - ◇ the equation of state,  $p\nu = RT$
  - ◇ internal energy depends only on temperature,  $u = u(T)$

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- the basic assumptions underlying the ideal gas model are
  - ◇ the equation of state,  $p\nu = RT$
  - ◇ internal energy depends only on temperature,  $u = u(T)$
- it follows that  $h$ ,  $c_v$  and  $c_p$  also depend only on temperature
- so specific internal energy and enthalpy differentials are

$$du = c_v(T)dT$$

$$dh = c_p(T)dT$$

## Partial derivatives from 1st $TdS$ equation

- the 1st  $TdS$  equation (per unit mass) is

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- so

$$\left( \frac{\partial s}{\partial T} \right)_v = \frac{c_v(T)}{T}, \quad \left( \frac{\partial s}{\partial v} \right)_T = \frac{R}{v}$$



## Ideal gas entropy changes from 1st $TdS$ equation

- applying our integration formula to  $s = s(T, v)$ :

$$s_2 - s_1 = \int_{T_1}^{T_2} \left( \frac{\partial s}{\partial T} \right)_{v=v_1} dT + \int_{v_1}^{v_2} \left( \frac{\partial s}{\partial v} \right)_{T=T_2} dv$$

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## Ideal gas entropy changes from 2nd $TdS$ equation

- we can apply the same procedure to the 2nd  $TdS$  equation
- it establishes that the partial derivatives of  $s = s(T, p)$  are

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p(T)}{T}, \quad \left(\frac{\partial s}{\partial p}\right)_T = -\frac{R}{p}$$

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- so our integration formula for  $s = s(T, p)$  gives

$$s_2 - s_1 = \int_{T_1}^{T_2} \frac{c_p(T)}{T} dT - R \ln \left( \frac{p_2}{p_1} \right)$$

## Ideal gas entropy changes with constant specific heats

- if  $c_v(T) \approx c_v^{\text{av}}$  between  $T_1$  and  $T_2$ , then

$$\int_{T_1}^{T_2} \frac{c_v(T)}{T} dT \approx c_v^{\text{av}} \ln \left( \frac{T_2}{T_1} \right)$$

so

$$s_2 - s_1 \approx c_v^{\text{av}} \ln \left( \frac{T_2}{T_1} \right) + R \ln \left( \frac{v_2}{v_1} \right)$$

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- similarly, if  $c_p(T) \approx c_p^{\text{av}}$  between  $T_1$  and  $T_2$ , then

$$s_2 - s_1 \approx c_p^{\text{av}} \ln \left( \frac{T_2}{T_1} \right) - R \ln \left( \frac{p_2}{p_1} \right)$$

## Ideal gas entropy changes with varying specific heats

- when specific heats vary significantly, we use ideal gas tables
- these tables have data for

$$s^{\circ}(T) = \int_{T_0}^T \frac{c_p(T)}{T} dT$$

- $T_0$  is an arbitrarily-chosen reference temperature



## Why tabulate $s^\circ(T)$ ?

- the change in  $s^\circ(T)$  between  $T_1$  and  $T_2$  is

$$\begin{aligned} s^\circ(T_2) - s^\circ(T_1) &= \int_{T_0}^{T_2} \frac{c_p(T)}{T} dT - \int_{T_0}^{T_1} \frac{c_p(T)}{T} dT \\ &= \int_{T_1}^{T_2} \frac{c_p(T)}{T} dT \end{aligned}$$

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- this is exactly the integral in our 2nd  $\Delta s$  formula:

$$\begin{aligned}s_2 - s_1 &= \int_{T_1}^{T_2} \frac{c_p(T)}{T} dT - R \ln \left( \frac{p_2}{p_1} \right) \\ &= s^\circ(T_2) - s^\circ(T_1) - R \ln \left( \frac{p_2}{p_1} \right)\end{aligned}$$

- if we know states 1 and 2, we can find all data on the RHS

## Summary of ideal gas entropy change formulas

- if specific heats are  $\sim$ constant, then

$$s_2 - s_1 \approx c_v^{\text{av}} \ln \left( \frac{T_2}{T_1} \right) + R \ln \left( \frac{v_2}{v_1} \right)$$

$$s_2 - s_1 \approx c_p^{\text{av}} \ln \left( \frac{T_2}{T_1} \right) - R \ln \left( \frac{p_2}{p_1} \right)$$

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(use whichever is more convenient)

- if specific heats vary significantly, then

$$s_2 - s_1 = s^\circ(T_2) - s^\circ(T_1) - R \ln \left( \frac{p_2}{p_1} \right)$$

(look up  $s^\circ(T_1)$  and  $s^\circ(T_2)$  in ideal gas table)

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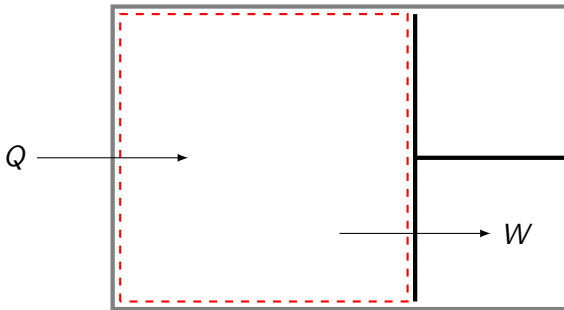
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Example

## Problem statement

A closed piston-cylinder device contains 1 kg of air, initially at 300 K, that expands at a constant 100 kPa to 0.746 m<sup>3</sup>. Calculate the air's specific entropy change (a) using an ideal gas table and (b) assuming constant specific heats.

# System diagram



# Given and find

- **given:**

- ◇  $m = 1 \text{ kg}$
- ◇  $p = 100 \text{ kPa}$
- ◇  $T_1 = 300 \text{ K}$
- ◇  $V_2 = 0.746 \text{ m}^3$

- **find:**

- (a)  $\Delta s$  using an ideal gas table
- (b)  $\Delta s$  assuming constant specific heats



# Assumptions and basic equations

- **assume:**

- ◇ closed system
- ◇ ideal gas
- ◇ constant pressure

- **basic equations:**

- ◇  $p\nu = RT$
- ◇  $\Delta s = s^\circ(T_2) - s^\circ(T_1) - R \ln(p_2/p_1)$
- ◇  $\Delta s \approx c_p^{\text{av}} \ln(T_2/T_1) - R \ln(p_2/p_1)$

## Solution to part (a)

- ideal gas, constant pressure:  $pv_2 = RT_2$ , so

$$T_2 = \frac{pv_2}{R} = \frac{(100\text{kPa})(0.746\text{m}^3/\text{kg})}{0.287\text{kJ}/\text{kg}/\text{K}} = 260\text{K}$$

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- ideal gas table:
  - ◇  $s^\circ(300\text{ K}) = 1.703\text{ kJ/kg/K}$
  - ◇  $s^\circ(260\text{ K}) = 1.559\text{ kJ/kg/K}$

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- ideal gas table:
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  - ◇  $s^\circ(260\text{ K}) = 1.559\text{ kJ/kg/K}$
- so change in specific entropy is

$$\begin{aligned} s_2 - s_1 &= s^\circ(T_2) - s^\circ(T_1) - R \ln \left( \frac{p_2}{p_1} \right) \\ &= (1.559 - 1.703)\text{kJ/kg/K} \\ &= -0.144\text{kJ/kg/K} \end{aligned}$$

## Solution to part (b)

- specific heat table:  $c_p(275 \text{ K}) = 1.004 \text{ kJ/kg/K}$ , so

$$\begin{aligned} s_2 - s_1 &\approx c_p^{\text{av}} \ln \left( \frac{T_2}{T_1} \right) - R \ln \left( \frac{p_2}{p_1} \right) \\ &= (1.004 \text{ kJ/kg/K}) \ln \left( \frac{260 \text{ K}}{300 \text{ K}} \right) \\ &= -0.1437 \text{ kJ/kg/K} \end{aligned}$$