# Lecture 30 – Isentropic processes

Purdue ME 200, Thermodynamics I

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# Outline

Isentropic processes

Ideal gases in general

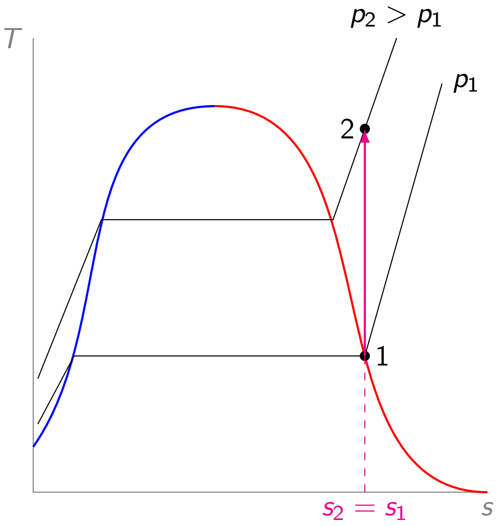
Ideal gases with constant specific heats

Example

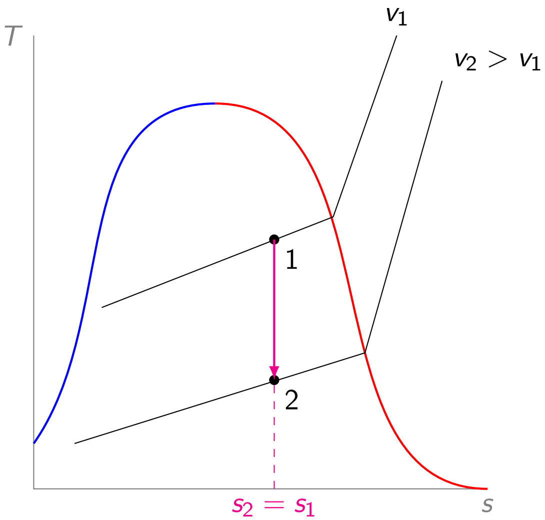
# Isentropic processes

- **isentropic** means constant entropy, $S_2 = S_1$
- if $m_2 = m_1$ in an isentropic process, then $s_2 = s_1$
- this is true for closed systems
- it's also true for steady-flow and steady-state processes
- a bit loosely, we often call any process with $s_2 = s_1$ isentropic

# Isentropic processes on $T$-$s$ diagrams (with isobars)



# Isentropic processes on $T$-$s$ diagrams (with isochors)



# Outline

# Reminder: entropy changes for ideal gases

- the entropy change for an ideal gas over any process is

$$s_2 - s_1 = s^\circ(T_2) - s^\circ(T_1) - R \ln\left(\frac{p_2}{p_1}\right)$$

- this is true whether or not specific heats are constant
- $s^\circ(T_1)$ and $s^\circ(T_2)$ live in ideal gas tables

# Entropy changes for ideal gases in isentropic processes

- for an ideal gas in an isentropic process,

$$s_2 = s_1 \implies s^\circ(T_2) - s^\circ(T_1) = R \ln\left(\frac{p_2}{p_1}\right)$$

- equivalently,

$$\begin{aligned}\frac{p_2}{p_1} &= e^{(s^\circ(T_2) - s^\circ(T_1))/R} \\ &= e^{s^\circ(T_2)/R} e^{-s^\circ(T_1)/R} \\ &= \frac{e^{s^\circ(T_2)/R}}{e^{s^\circ(T_1)/R}}\end{aligned}$$

# Relative pressure (tabulated for air)

- define the **relative pressure**

$$p_r(T) = e^{s^\circ(T)/R}$$

- then for an ideal gas in an isentropic process,

$$\frac{p_2}{p_1} = \frac{e^{s^\circ(T_2)/R}}{e^{s^\circ(T_1)/R}} = \frac{p_r(T_2)}{p_r(T_1)}$$

- for air, $p_r(T_1)$ and $p_r(T_2)$ can be found in ideal gas tables

# Relative volume (tabulated for air)

- define the **relative volume**

$$v_r(T) = \frac{RT}{p_r(T)}$$

- then for an ideal gas in an isentropic process,

$$\frac{v_2}{v_1} = \frac{RT_2/p_2}{RT_1/p_1} = \frac{RT_2}{RT_1}\frac{p_1}{p_2}$$

- but $p_1/p_2 = p_r(T_1)/p_r(T_2)$, so

$$\frac{v_2}{v_1} = \frac{RT_2}{RT_1}\frac{p_r(T_1)}{p_r(T_2)} = \frac{RT_2/p_r(T_2)}{RT_1/p_r(T_1)} = \frac{v_r(T_2)}{v_r(T_1)}$$

- for air, $v_r(T_1)$ and $v_r(T_2)$ can be found in ideal gas tables

# Summary

- for any ideal gas in an isentropic process,

$$\frac{p_2}{p_1} = \frac{e^{s^\circ(T_2)/R}}{e^{s^\circ(T_1)/R}}$$

  (all ideal gas tables include $s^\circ(T)$)

- for air, it can save time to use

$$\frac{p_2}{p_1} = \frac{p_r(T_2)}{p_r(T_1)} \quad \text{or} \quad \frac{v_2}{v_1} = \frac{v_r(T_2)}{v_r(T_1)}$$

  (ideal gas table for air also includes $p_r(T)$ and $v_r(T)$)

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# Reminder: ideal gas entropy changes with constant $c_v$, $c_p$

- in any process where ideal gas specific heats are ~constant,

$$s_2 - s_1 \approx c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{v_2}{v_1}\right)$$
$$s_2 - s_1 \approx c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

- so for an ideal gas in an isentropic process,

$$c_v \ln\left(\frac{T_2}{T_1}\right) \approx -R \ln\left(\frac{v_2}{v_1}\right)$$
$$c_p \ln\left(\frac{T_2}{T_1}\right) \approx R \ln\left(\frac{p_2}{p_1}\right)$$

# Two expressions for the temperature ratio

- rearranging the first equation,

$$\ln\left(\frac{T_2}{T_1}\right) \approx -\frac{R}{c_v}\ln\left(\frac{v_2}{v_1}\right) = \ln\left[\left(\frac{v_2}{v_1}\right)^{-R/c_v}\right]$$

$$\implies \frac{T_2}{T_1} \approx \left(\frac{v_2}{v_1}\right)^{-R/c_v}$$

- rearranging the second equation,

$$\ln\left(\frac{T_2}{T_1}\right) \approx \frac{R}{c_p}\ln\left(\frac{p_2}{p_1}\right) = \ln\left[\left(\frac{p_2}{p_1}\right)^{R/c_p}\right]$$

$$\implies \frac{T_2}{T_1} \approx \left(\frac{p_2}{p_1}\right)^{R/c_p}$$

# Reminder: ideal gases specific heats

- ideal gas specific heats always satisfy $c_p(T) = c_v(T) + R$
- in terms of the specific heat ratio $k(T) = c_p(T)/c_v(T)$,

$$\frac{R}{c_v(T)} = k(T) - 1 \quad \text{and} \quad \frac{R}{c_p(T)} = \frac{k(T) - 1}{k(T)}$$

- so the two expressions for $T_2/T_1$ can be written as

$$\frac{T_2}{T_1} \approx \left(\frac{v_2}{v_1}\right)^{-R/c_v} = \left(\frac{v_2}{v_1}\right)^{-(k-1)} = \left(\frac{v_1}{v_2}\right)^{k-1}$$

$$\frac{T_2}{T_1} \approx \left(\frac{p_2}{p_1}\right)^{R/c_p} = \left(\frac{p_2}{p_1}\right)^{(k-1)/k}$$

# Ideal gas + isentropic + constant $c_v$ & $c_p$ = polytropic

- equating the two expressions for $T_2/T_1$ gives

$$\left(\frac{p_2}{p_1}\right)^{(k-1)/k} \approx \left(\frac{v_1}{v_2}\right)^{k-1} \iff \frac{p_2}{p_1} \approx \left(\frac{v_1}{v_2}\right)^{k}$$

- so $p_2 v_2^k \approx p_1 v_1^k$

$\implies$ the process is ~polytropic with coefficient $k = c_p/c_v$

# Summary

- for ideal gas in isentropic process with ~constant $k = c_p/c_v$,

$$p_1 v_1^k \approx p_2 v_2^k$$

- also,

$$\frac{T_2}{T_1} \approx \left(\frac{p_2}{p_1}\right)^{(k-1)/k} \approx \left(\frac{v_1}{v_2}\right)^{k-1}$$

- and (from lecture 12) boundary work is

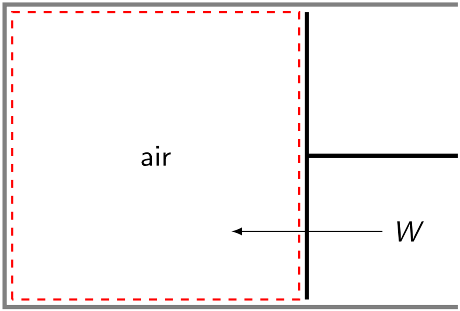
$$\int_{V_1}^{V_2} p \mathrm{d}V \approx \frac{mR(T_2 - T_1)}{1 - k}$$

# Outline

# Problem statement

A closed insulated reversible piston-cylinder device compresses air from 22 $^\circ$C and 95 kPa to $1/8$ its initial volume. Find the final air temperature (a) assuming constant specific heats and (b) using an ideal gas table.

# System diagram



# Given and find

- **given:**
  - ◇ $p_1 = 95$ kPa
  - ◇ $T_1 = 22$ °C $= 295$ K
  - ◇ $V_2 = V_1/8$
- **find:**
  - (a) $T_2$ assuming constant specific heats
  - (b) $T_2$ using an ideal gas table

# Assumptions and basic equations

- **assume:**
  - ◇ closed system ($m_2 = m_1$)
  - ◇ ideal gas
  - ◇ isentropic (insulated + reversible) compression
- **basic equations:**
  - ◇ ideal gas with ∼constant specific heats in isentropic process:

$$\frac{T_2}{T_1} \approx \left(\frac{v_1}{v_2}\right)^{k-1} \quad \text{where} \quad k = \frac{c_p}{c_v}$$

  - ◇ air (as an ideal gas) in isentropic process:

$$\frac{v_2}{v_1} = \frac{v_r(T_2)}{v_r(T_1)}$$

# Solution to part (a)

- ideal gas with constant specific heats in isentropic process:

$$\frac{T_2}{T_1} \approx \left(\frac{v_1}{v_2}\right)^{k-1} \implies T_2 \approx T_1 \left(\frac{v_1}{v_2}\right)^{k-1}$$

- since $m_2 = m_1$,

$$\frac{v_2}{v_1} = \frac{mV_2}{mV_1} = \frac{V_2}{V_1} = \frac{1}{8}$$

- so

$$T_2 \approx 8^{k-1} T_1$$

# Solution to part (a) (continued)

- what value of $k = c_p/c_v$ to use?
- in reality, $c_p(T)$ and $c_v(T)$ vary with $T$, so $k(T)$ does too
- but we are asked to assume constant specific heats
- in these approximations, we usually use $T_{\text{av}} = (T_1 + T_2)/2$
- but in this calculation, $T_2$ is unknown
- so let's start with what we know: $k(T_1) = k(295\text{ K})$

# Solution to part (a) (continued)

- specific heat table: $c_p(295\text{ K}) = 1.005$ kJ/kg/K, so

$$\begin{aligned} c_v(295\text{K}) &= c_p(295\text{K}) - R \\ &= (1.005 - 0.287)\text{kJ/kg/K} \\ &= 0.718\text{kJ/kg/K} \end{aligned}$$

- so $k(295\text{K}) = 1.005/0.718 = 1.400$
- therefore, a tentative estimate of $T_2$ is

$$\hat{T}_2 = 8^{k(295\text{K})-1} T_1 = 8^{1.400-1}(295\text{K}) = 677.4\text{K}$$

# Solution to part (a) (continued)

- now we have an estimate $\hat{T}_2$ of $T_2$
- so we can estimate $T_{\text{av}}$ by

$$\hat{T}_{\text{av}} = \frac{T_1 + \hat{T}_2}{2} = \frac{295\text{K} + 677.4\text{K}}{2} = 486.2\text{K}$$

- repeating the table look-up and calculations, $k(\hat{T}_{\text{av}}) \approx 1.389$
- this gives a refined estimate of $T_2$:

$$T_2 \approx 8^{k(\hat{T}_{\text{av}})-1} T_1 = 8^{1.389-1}(295\text{K}) = 662.3\text{K}$$

# Solution to part (b)

- from the relative volume equation,

$$\frac{v_2}{v_1} = \frac{v_r(T_2)}{v_r(T_1)} \implies v_r(T_2) = v_r(T_1)\frac{v_2}{v_1}$$

- ideal gas table: at $T_1 = 295$ K, $v_r(T_1) = 647.9$, so

$$v_r(T_2) = v_r(T_1)\frac{v_2}{v_1} = \frac{647.9}{8} = 80.99$$

- ideal gas table: $v_r(660\text{K}) = 81.89$ and $v_r(670\text{K}) = 78.61$
- interpolating gives $T_2 = 662.7$ K