Thermal modeling of buildings

Purdue ME 597, Distributed Energy Resources

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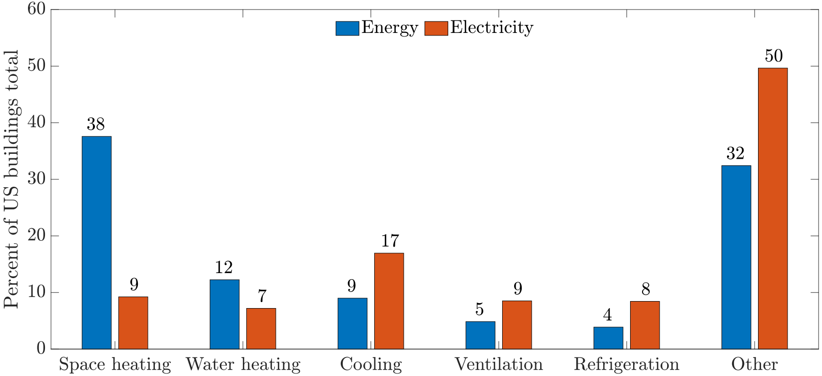
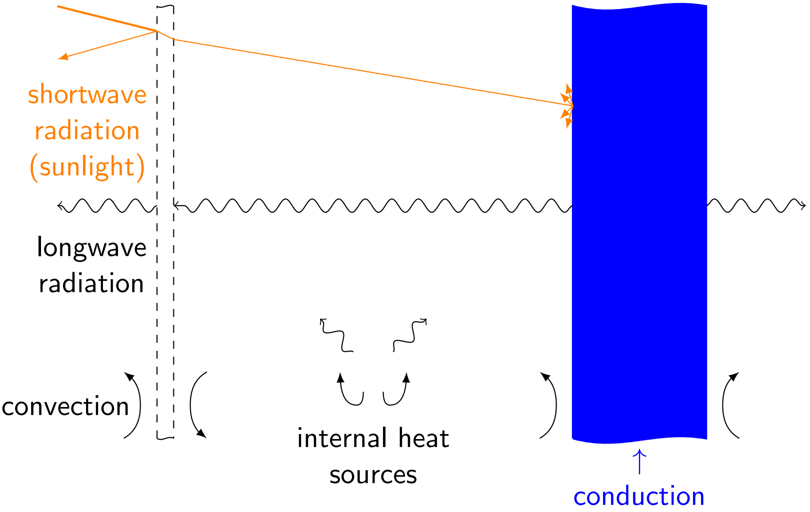
Outline

HVAC&R dominates building energy and electricity use



Detailed thermal modeling is hard

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \ldots$$



Software for detailed thermal modeling

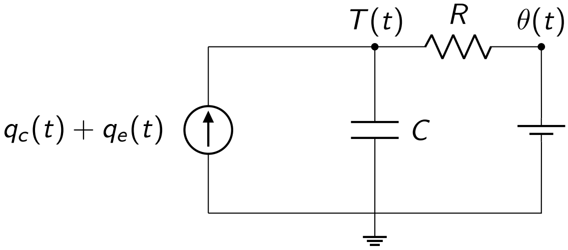
- EnergyPlus (US DOE, open source)
- TRNSYS (**Tr**a**n**sient **Sy**stem **S**imulation, commercial)
- Modelica buildings library (US DOE, open source)
- these programs are broad and powerful, but can
  - ◇ require many parameters, some hard to specify or fit
  - ◇ be slow (numerical solution of coupled nonlinear PDEs)
  - ◇ be difficult to integrate with other software
- our focus: simpler models for DER design and control

Outline

Thermal circuit models capture dominant physics

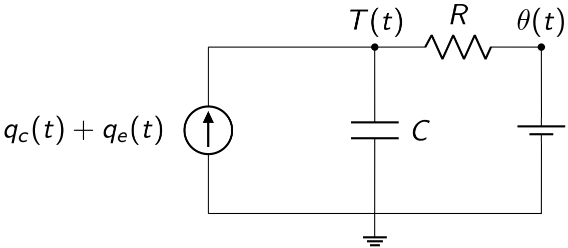
- thermal circuits work by analogy to electrical circuits
  - ⋄ temperature $\leftrightarrow$ voltage
  - ⋄ heat $\leftrightarrow$ charge
  - ⋄ thermal resistance $\leftrightarrow$ electrical resistance
  - ⋄ thermal capacitance $\leftrightarrow$ electrical capacitance
- basic idea: temperature differences drive heat flow
  (like voltage differences drive charge flow [current])

Simplest thermal circuit: 1R1C



- $T(t)$ (°C) is indoor temperature
- $\theta(t)$ (°C) is boundary (often outdoor air) temperature
- R (°C/kW) is thermal resistance between T and θ
- C (kWh/°C) is indoor thermal capacitance
- $q_c(t)$ (kW) is thermal power from controlled equipment
- $q_e(t)$ (kW) is thermal power from exogenous sources (sunlight, body heat, plug loads, ...)

1R1C continuous-time dynamics



- Ohm's law: current through resistor is $(T(t) - \theta(t))/R$
- rate of charge accumulation on capacitor is $C\mathrm{d}T(t)/\mathrm{d}t$
- Kirchhoff's Current Law (KCL) at node labeled $T(t)$:

$$\text{current inflow} = \text{current outflow}$$

$$q_c(t) + q_e(t) = \frac{T(t) - \theta(t)}{R} + C\frac{\mathrm{d}T(t)}{\mathrm{d}t}$$

- rearranging continuous-time dynamics gives

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{1}{RC}\left[-T(t) + \theta(t) + R\left(q_c(t) + q_e(t)\right)\right],$$

  a first-order linear ODE

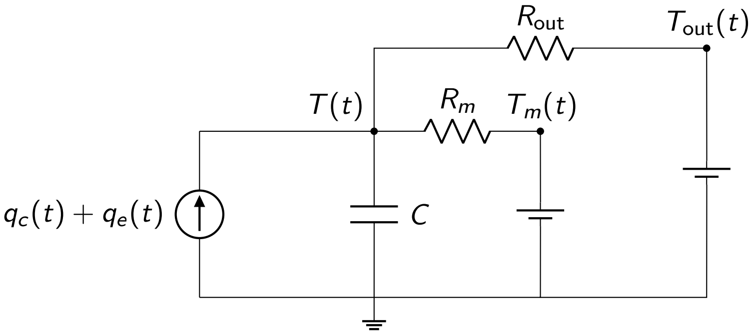
- with uniform time step Δt and piecewise constant θ, q_c, q_e,

$$T(k+1) = aT(k) + (1-a)R\left(q_c(k) + w(k)\right)$$

  where $a = e^{-\Delta t/(RC)}$ and $w(k) = q_e(k) + \theta(k)/R$

- **homework:** show this

A 2R1C thermal circuit



- $T_m(t)$ (°C) is thermal mass temperature
- $T_{\text{out}}(t)$ (°C) is outdoor air temperature
- R_m (°C/kW) is thermal resistance between T and T_m
- R_{out} (°C/kW) is thermal resistance between T and T_{out}

Reducing a 2R1C thermal circuit to 1R1C

- KCL at node labeled $T(t)$ gives 2R1C dynamics:

$$C\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{T_m(t) - T(t)}{R_m} + \frac{T_{\text{out}}(t) - T(t)}{R_{\text{out}}} + q_c(t) + q_e(t)$$

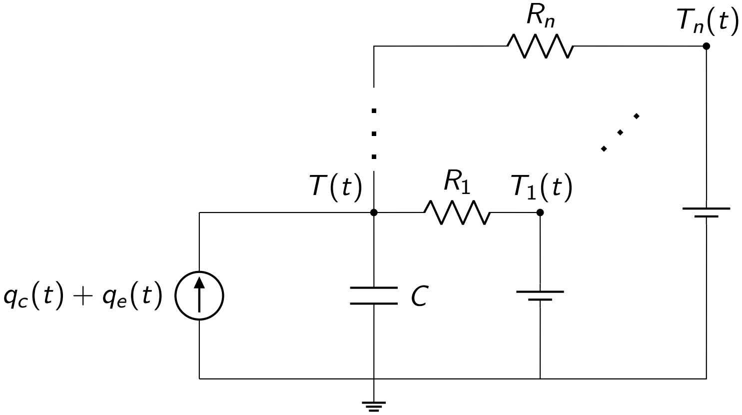
- **homework:** show this is equivalent to the 1R1C model

$$C\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{\theta(t) - T(t)}{R} + q_c(t) + q_e(t)$$

with

$$R = \frac{R_m R_{\text{out}}}{R_m + R_{\text{out}}}$$
$$\theta(t) = \frac{R_{\text{out}} T_m(t) + R_m T_{\text{out}}(t)}{R_m + R_{\text{out}}}$$

An nR1C thermal circuit



$$\text{KCL} \implies C\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{T_i(t) - T(t)}{R_i} + q_c(t) + q_e(t)$$

Reducing an nR1C thermal circuit to 1R1C

- nR1C model is equivalent to 1R1C model

$$C\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{\theta(t) - T(t)}{R} + q_c(t) + q_e(t)$$

with

$$R = \frac{R_1 \cdots R_n}{R_2 \cdots R_n + \cdots + R_1 \cdots R_{n-1}}$$
$$\theta(t) = \frac{R_2 \cdots R_n T_1(t) + \cdots + R_1 \cdots R_{n-1} T_n(t)}{R_2 \cdots R_n + \cdots + R_1 \cdots R_{n-1}}$$

- **homework:** show this

Outline

Thermal circuit parameters

thermal circuit models have three types of parameter:

1. thermal resistance $R > 0$ ($^\circ$C/kW)
2. thermal capacitance $C > 0$ (kWh/$^\circ$C)
3. exogenous thermal power $q_e(t)$ (kW, a time series)

Typical thermal resistance values

- can model thermal resistance as $R = 1/(UA_w)$, where
  - A_w (m^2) is exterior walls' outward-facing surface area
  - U (kW/[$^\circ$C m^2]) is their overall thermal transmittance
- typical thermal transmittances (note units of **W**, not **kW**) are
  - $\sim$1 to 6 **W**/($^\circ$C m^2) for triple- to single-pane windows
  - $\sim$0.2 to 0.8 **W**/($^\circ$C m^2) for framed, insulated, opaque walls

A typical wall assembly

The Gold Hive: How (and why) to install rockwool insulation

Typical thermal resistance values (continued)

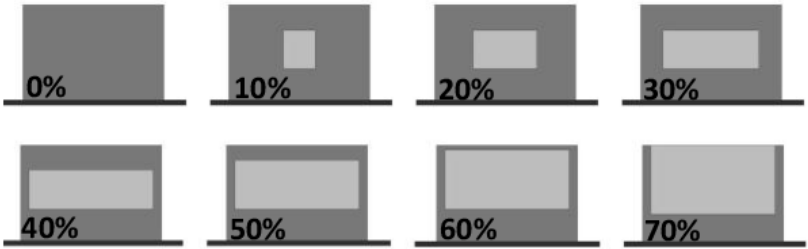
- for a wall assembly whose surface area is $100\lambda\%$ windows,

$$U = \lambda U_{\text{window}} + (1-\lambda)U_{\text{wall}}$$

- a typical wall assembly has $\lambda \approx 0.2$ to 0.4

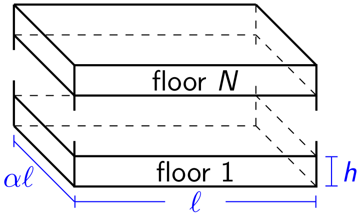
$\Longrightarrow$ overall $U \approx 0.9$ to 2 $\mathbf{W}/({}^\circ\text{C m}^2)$

Window-to-wall ratio λ



El-Deeb (2013): Combined effect of window-to-wall ratio and wall composition on energy consumption

Typical thermal resistance values (continued)



floor aspect ratio: α

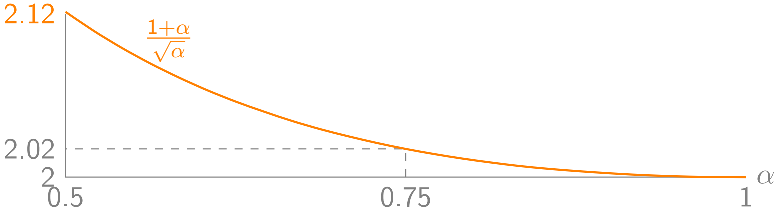
total floor area: $A_f = \alpha N \ell^2$

$\implies \ell = \sqrt{A_f/(\alpha N)}$

total wall area: $A_w = 2(1+\alpha) N \ell h$

$\implies A_w = 2(1+\alpha) h \sqrt{N A_f/\alpha}$

$$\frac{1+\alpha}{\sqrt{\alpha}} \approx 2, \text{ so } A_w \approx 4h\sqrt{NA_f}$$



Typical thermal capacitance values

- can model thermal capacitance as $C = \rho c_p V$, where
  - $\rho = 1.293$ kg/m^3
  - $c_p = 2.792 \times 10^{-4}$ kWh/(kg °C)
  - $V = A_f h$ is enclosed volume
- this models air at standard temperature and pressure
- often, other stuff (ducts, etc.) is tightly coupled to indoor air

$$\begin{aligned} \Longrightarrow \; C &\approx (10 \text{ to } 15)\rho c_p V \\ &\approx (0.011 \text{ to } 0.014 \text{ kWh/}[^\circ\text{C m}^2])A_f \end{aligned}$$

Summary: Typical resistance and capacitance values

- for a building with N stories and total floor area A_f,

$$R \approx \frac{1}{(0.01 \text{ to } 0.022 \text{ kW}/[^\circ\text{C m}])\sqrt{NA_f}}$$
$$C \approx (0.011 \text{ to } 0.014 \text{ kWh}/[^\circ\text{C m}^2])A_f$$

- time constant: $\tau = RC \approx (0.6 \text{ to } 1.3)\sqrt{A_f/N}$
- plausible values for a 200 m^2 house with 2 stories:

$$R \approx 3.5\ ^\circ\text{C/kW}, \;\; C \approx 2.5\ \text{kWh}/^\circ\text{C}, \;\; \tau \approx 8.5\ \text{h}$$

Typical exogenous thermal power values

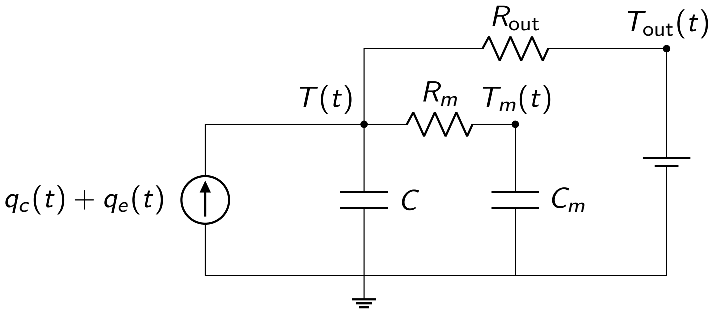
- can decompose $q_e(t)$ into heat transfer from
  1. plugged-in devices (lights, electronics, electric cooking, ...)
  2. sun
  3. other (body heat, fossil-fueled cooking, wood fireplaces, ...)
- often, #1 is measured via input electricity
  (typical intensity for homes: 5 to 10 W per m^2 of floor area)
- for #2, can use solar irradiance data $I(t)$ (kW/m^2):

$$q_{\text{sun}}(t) \approx (0.15 \text{ to } 0.23 \text{ m})\sqrt{NA_f}\, I(t)$$

  (more on this when we learn about solar)
- #3 is often small (e.g., ∼100 W per body)

Outline

A 2R2C thermal circuit



$$C \frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{T_m(t) - T(t)}{R_m} + \frac{T_{\text{out}}(t) - T(t)}{R_{\text{out}}} + q_c(t) + q_e(t)$$

$$C_m \frac{\mathrm{d}T_m(t)}{\mathrm{d}t} = \frac{T(t) - T_m(t)}{R_m}$$

Discrete-time dynamics

- continuous-time dynamics are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} T(t) \\ T_m(t) \end{bmatrix} = \tilde{A}\begin{bmatrix} T(t) \\ T_m(t) \end{bmatrix} + \tilde{B}(q_c(t) + w(t))$$

  with

$$\tilde{A} = \begin{bmatrix} -(1/R_m + 1/R_{\text{out}})/C & 1/(R_m C) \\ 1/(R_m C_m) & -1/(R_m C_m) \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 1/C \\ \ \end{bmatrix}, \ w(t) = q_e(t) + T_{\text{out}}(t)/R_{\text{out}}$$

- can exactly discretize this LDS via matrix exponential to

$$\begin{bmatrix} T(k+1) \\ T_m(k+1) \end{bmatrix} = A\begin{bmatrix} T(k) \\ T_m(k) \end{bmatrix} + B\left(q_c(k) + w(k)\right)$$

  where $A = e^{\Delta t \tilde{A}}$, $B = (A - I)\tilde{A}^{-1}\tilde{B}$ (can show $\tilde{A}^{-1}$ exists)

Thermal mass parameter values

- in empirical studies of real buildings, typically
  - $C_m \approx (8 \text{ to } 16)C$
  - $R_m \approx R/(4 \text{ to } 8)$
- check: $C_m/(0.3 \text{ kWh}/[\text{m}^3\ ^\circ\text{C}]) \approx$ equivalent volume of pine

Penman (1990): Second order system identification in the thermal response of a working school

Two-timing

- T typically changes much faster than T_m

$\Longrightarrow$ air and mass dynamics define two characteristic time scales

- fast time scale: $T_m(t) \approx T_{m0}$ (a constant) for all t
- slow time scale: $\mathrm{d}T(t)/\mathrm{d}t \approx 0$ for (almost) all t

Fast dynamics: 1R1C with state $T(t)$

$$
\begin{aligned}
C\frac{\mathrm{d}T(t)}{\mathrm{d}t} &\approx \frac{T_{m0} - T(t)}{R_m} + \frac{T_{\text{out}}(t) - T(t)}{R_{\text{out}}} + q_c(t) + q_e(t) \\
&= \frac{\theta(t) - T(t)}{R} + q_c(t) + q_e(t)
\end{aligned}
$$

with

$$
\begin{aligned}
R &= \frac{R_m R_{\text{out}}}{R_m + R_{\text{out}}} \\
\theta(t) &= \frac{R_{\text{out}} T_{m0} + R_m T_{\text{out}}(t)}{R_m + R_{\text{out}}}
\end{aligned}
$$

Slow dynamics: 1R1C with state $T_m(t)$

$$C_m \frac{\mathrm{d}T_m(t)}{\mathrm{d}t} = \frac{T(t) - T_m(t)}{R_m}$$

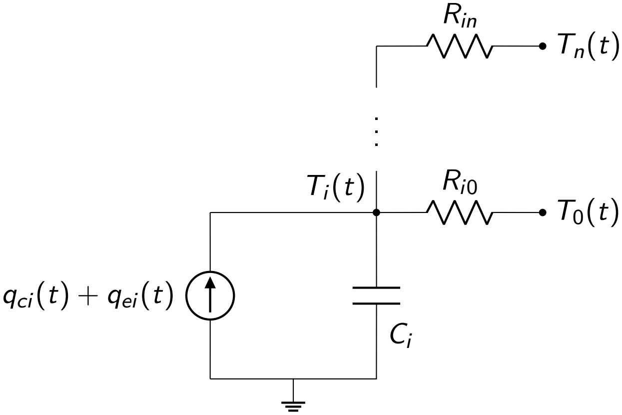
- if $T(t)$ is ~constant over each time step of duration Δt, then

  $$T_m(k+1) \approx a_m T_m(k) + (1 - a_m) T(k)$$

  where $a_m = e^{-\Delta t/(R_m C_m)}$
- since $\mathrm{d}T(t)/\mathrm{d}t \approx 0$,

  $$q_c(t) \approx \frac{T(t) - T_m(t)}{R_m} + \frac{T(t) - T_{\text{out}}(t)}{R_{\text{out}}} - q_e(t)$$

General mRnC thermal circuits



$$C_i \frac{\mathrm{d}T_i(t)}{\mathrm{d}t} = \sum_{j=0}^{n} \frac{T_j(t) - T_i(t)}{R_{ij}} + q_{ci}(t) + q_{ei}(t)$$

General mRnC thermal circuits (continued)

- node 0 is a boundary node (such as the outdoor air)
- $1/R_{ij} = 0$ if no heat transfers between nodes i and j
- $q_{ci} = 0$ if equipment transfers no heat to or from node i
- can put mRnC model in matrix form and discretize like 2R2C

Outline

Perfect setpoint tracking control (1R1C)

- define indoor temperature setpoints $\hat{T}(k)$
- 1R1C discrete-time dynamics are

$$T(k+1) = aT(k) + (1-a)R(q_c(k) + w(k))$$

$\implies$ to drive temperature from $T(k)$ to $T(k+1) = \hat{T}(k+1)$, set

$$\hat{q}_c(k) = \frac{\hat{T}(k+1) - aT(k)}{(1-a)R} - w(k)$$

Perfect setpoint tracking control (2R2C)

- 2R2C discrete-time dynamics are

$$\begin{bmatrix} T(k+1) \\ T_m(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} T(k) \\ T_m(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} (q_c(k) + w(k))$$

$\implies$ to drive temperature from $T(k)$ to $T(k+1) = \hat{T}(k+1)$, set

$$\hat{q}_c(k) = \frac{\hat{T}(k+1) - A_{11} T(k) - A_{12} T_m(k)}{B_1} - w(k)$$

- can treat arbitrary mRnC thermal circuits similarly

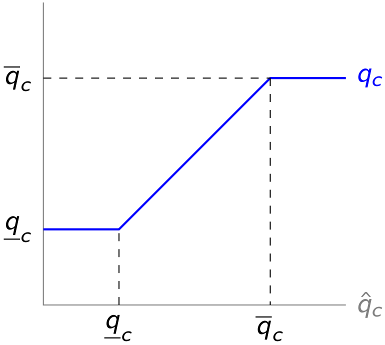
(Near-)perfect setpoint tracking with capacity limits

- heating and cooling systems have capacity constraints

$$\underline{q}_c(k) \leq q_c(k) \leq \overline{q}_c(k)$$

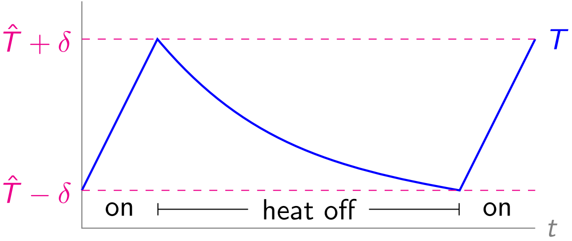
- to respect capacity constraints, saturate $\hat{q}_c(k)$:

$$q_c(k) = \max\left\{\underline{q}_c(k), \min\left\{\overline{q}_c(k), \hat{q}_c(k)\right\}\right\}$$



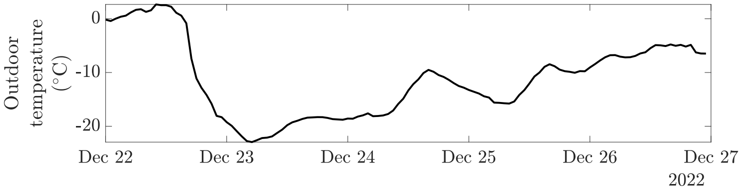
Thermostatic control

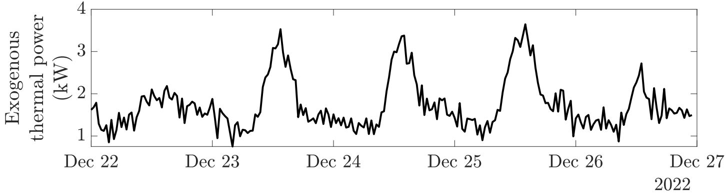
- many heating and cooling systems operate in an on/off fashion
- these systems typically use thermostatic control
- heating example with action $u(k) \in \{0, 1\}$, deadband δ (°C):
  - ◇ initialize $u(k) = u(k-1)$
  - ◇ if $T(k) > \hat{T}(k) + \delta$, set $u(k) = 0$
  - ◇ if $T(k) < \hat{T}(k) - \delta$, set $u(k) = 1$
  - ◇ set $q_c(k) = \underline{q}_c(k) + u(k)\left(\overline{q}_c(k) - \underline{q}_c(k)\right)$
  - ◇ increment k, update $T(k)$, and repeat



Example: Exact 2R2C vs. two-timing

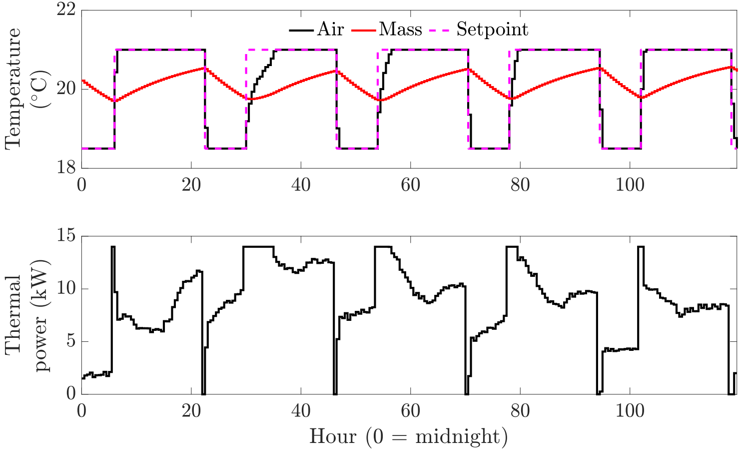
- $N = 2$ story house with $A_f = 200$ m^2 total floor area
- simulated over 5 very cold days from 2022 in West Lafayette





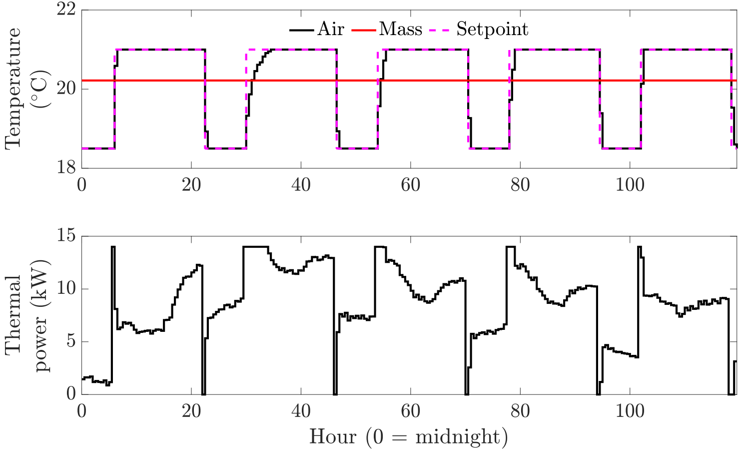
Exact 2R2C simulation results

- lower indoor air temperature setpoint overnight
- control tries to track setpoint but saturates at capacity limits



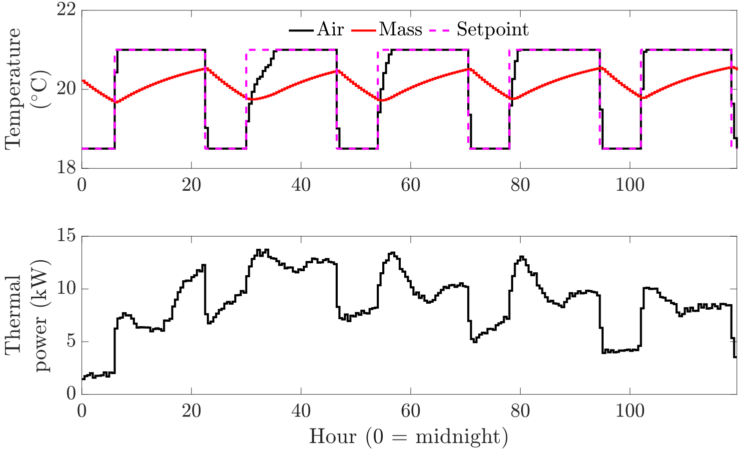
Fast 1R1C simulation results

- mass temperature assumed constant at time-average setpoint
- 0.36 kW thermal power MAE, 10.4 kWh (1%) energy error



Slow 1R1C simulation results

- quasi-steady thermal power model, air temperature from 2R2C
- 0.7 kW thermal power MAE, 1.2 kWh (0.1%) energy error



Outline

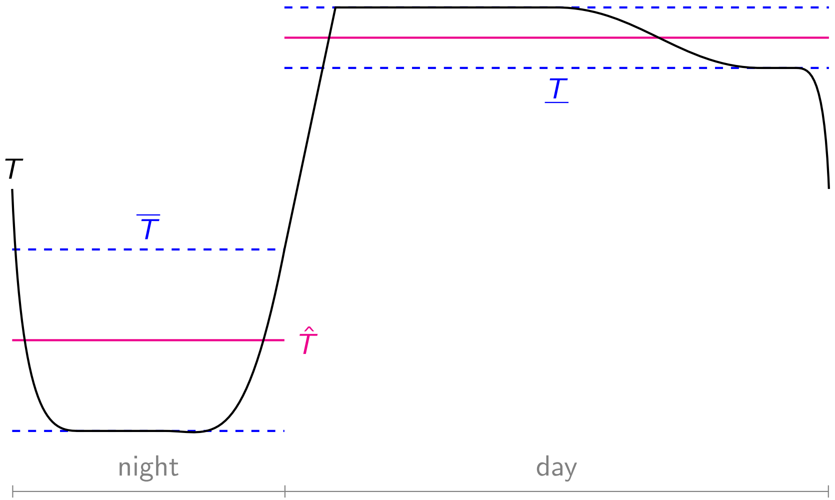
one battery model is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\frac{x(t)}{\tau} + p^{\text{chem}}(t)$$
$$\underline{x} \leq x(t) \leq \overline{x}$$
$$-\overline{p}_d^{\text{chem}} \leq p^{\text{chem}}(t) \leq \overline{p}_c^{\text{chem}}$$

Indoor air as a thermal battery

- at fast time scales, $C\frac{\mathrm{d}T(t)}{\mathrm{d}t} = \frac{\theta(t) - T(t)}{R} + q_c(t) + q_e(t)$
- define nominal $\hat{T}$, $\hat{q}_c$ with $C\frac{\mathrm{d}\hat{T}(t)}{\mathrm{d}t} = \frac{\theta(t) - \hat{T}(t)}{R} + \hat{q}_c(t) + q_e(t)$
- then thermal energy $x(t) = C(T(t) - \hat{T}(t))$ satisfies

$$\begin{aligned}
\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= C\left(\frac{\mathrm{d}T(t)}{\mathrm{d}t} - \frac{\mathrm{d}\hat{T}(t)}{\mathrm{d}t}\right) \\
&= \frac{\hat{T}(t) - T(t)}{R} + q_c(t) - \hat{q}_c(t) \\
&= -\underbrace{\frac{x(t)}{RC}}_{x(t)/\tau} + \underbrace{q_c(t) - \hat{q}_c(t)}_{p^{\mathrm{thrm}}(t)}
\end{aligned}$$



T
$\overline{T}$
$\underline{T}$
$\hat{T}$
night
day

Indoor air as a thermal battery: Energy capacity

- suppose indoor air temperature has comfort constraints

$$\underline{T}(t) \le T(t) \le \overline{T}(t)$$

- then thermal energy $x(t) = C(T(t) - \hat{T}(t))$ must satisfy

$$\underbrace{C(\underline{T}(t) - \hat{T}(t))}_{\underline{x}(t)} \le x(t) \le \underbrace{C(\overline{T}(t) - \hat{T}(t))}_{\overline{x}(t)}$$

- for a house with $C \approx 2.5$ kWh/°C, $\overline{T}(t) - \underline{T}(t) \approx 2$ °C,

$$\overline{x}(t) - \underline{x}(t) \approx 5 \text{ kWh}$$

Indoor air as a thermal battery: Power capacities

- heating and cooling systems have capacity constraints

$$\underline{q}_c(t) \leq q_c(t) \leq \overline{q}_c(t)$$

- so $p^{\text{thrm}}(t) = q_c(t) - \hat{q}_c(t)$ must satisfy

$$\underbrace{\underline{q}_c(t) - \hat{q}_c(t)}_{-\overline{p}_d^{\text{thrm}}(t)} \leq p^{\text{thrm}}(t) \leq \underbrace{\overline{q}_c(t) - \hat{q}_c(t)}_{\overline{p}_c^{\text{thrm}}(t)}$$

- for a typical US house,

$$\overline{p}_c^{\text{thrm}} + \overline{p}_d^{\text{thrm}} = \overline{q}_c(t) - \underline{q}_c(t) \approx 10 \text{ to } 15 \text{ kW}$$

Thermal mass as a thermal battery

- at slow time scales, $C_m \frac{\mathrm{d}T_m(t)}{\mathrm{d}t} = \frac{T(t) - T_m(t)}{R_m}$
- define nominal $\hat{T}$, $\hat{T}_m$ with $C_m \frac{\mathrm{d}\hat{T}_m(t)}{\mathrm{d}t} = \frac{\hat{T}(t) - \hat{T}_m(t)}{R_m}$
- then thermal energy $x_m(t) = C_m(T_m(t) - \hat{T}_m(t))$ satisfies

$$\begin{aligned}
\frac{\mathrm{d}x_m(t)}{\mathrm{d}t} &= C_m \left(\frac{\mathrm{d}T_m(t)}{\mathrm{d}t} - \frac{\mathrm{d}\hat{T}_m(t)}{\mathrm{d}t} \right) \\
&= \frac{T(t) - \hat{T}(t) - (T_m(t) - \hat{T}_m(t))}{R_m} \\
&= -\underbrace{\frac{x_m(t)}{R_m C_m}}_{x_m(t)/\tau_m} + \underbrace{\frac{T(t) - \hat{T}(t)}{R_m}}_{p_m^{\mathrm{thrm}}(t)}
\end{aligned}$$

Thermal mass as a thermal battery: Energy capacity

- suppose thermal mass temperature must satisfy

$$\underline{T}_m(t) \le T_m(t) \le \overline{T}_m(t)$$

- then thermal energy $x_m(t) = C(T_m(t) - \hat{T}_m(t))$ must satisfy

$$\underbrace{C_m(\underline{T}_m(t) - \hat{T}_m(t))}_{\underline{x}_m(t)} \le x_m(t) \le \underbrace{C_m(\overline{T}_m(t) - \hat{T}_m(t))}_{\overline{x}_m(t)}$$

- for a house with $C_m \approx 25$ kWh/$^\circ$C, $\overline{T}_m(t) - \underline{T}_m(t) \approx 2\ ^\circ$C,

$$\overline{x}_m(t) - \underline{x}_m(t) \approx 50 \text{ kWh}$$

Thermal mass as a thermal battery: Power capacities

- indoor air temperature has constraints

$$\underline{T}(t) \leq T(t) \leq \overline{T}(t)$$

- so $p_m^{\text{thrm}}(t) = (T(t) - \hat{T}(t))/R_m$ must satisfy

$$\underbrace{\frac{\underline{T}(t) - \hat{T}(t)}{R_m}}_{-\overline{p}_d^{\text{thrm}}(t)} \leq p_m^{\text{thrm}}(t) \leq \underbrace{\frac{\overline{T}(t) - \hat{T}(t)}{R_m}}_{\overline{p}_c^{\text{thrm}}(t)}$$

- for a house with $R_m \approx 0.5$ °C/kW, $\overline{T}(t) - \underline{T}(t) \approx 2$ °C,

$$\overline{p}_c^{\text{thrm}} + \overline{p}_d^{\text{thrm}} \approx 4 \text{ kW}$$

Summary: Buildings as thermal batteries

- can view a 2R2C model of a building as two thermal batteries
- indoor air and 'shallow' thermal mass has
  - time constant RC
  - thermal energy capacity $C(\overline{T}(t) - \underline{T}(t))$
  - thermal charging power capacity $\overline{q}_c(t) - \hat{q}_c(t)$
  - thermal discharging power capacity $\hat{q}_c(t) - \underline{q}_c(t)$
- 'deep' thermal mass has
  - time constant $R_m C_m$
  - thermal energy capacity $C_m(\overline{T}_m(t) - \underline{T}_m(t))$
  - thermal charging power capacity $(\overline{T}(t) - \hat{T}(t))/R_m$
  - thermal discharging power capacity $(\hat{T}(t) - \underline{T}(t))/R_m$

Comparing thermal and chemical energy storage

- a battery converts 1 kWh electrical back to ~1 kWh electrical
- a heat pump converts 1 kWh electrical to ~3 kWh thermal (more on heat pumps next lecture)
- so 1 kWh thermal storage is ‘worth’ ~1/3 kWh in a battery