Convex sets and functions

Purdue ME 597, Distributed Energy Resources

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these slides draw on materials by Stephen Boyd at Stanford

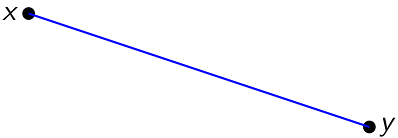
Outline

Line segments in $\mathbf{R}^n$

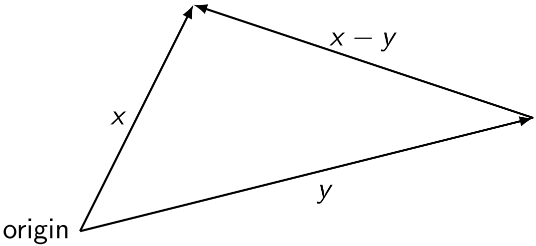
for $x,\ y \in \mathbf{R}^n$,

$$\{\theta x + (1-\theta) y \mid \theta \in [0,1]\}$$

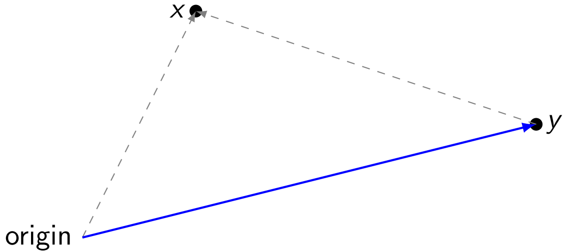
is the line segment connecting x and y



Line segments in $\mathbf{R}^n$ (continued)

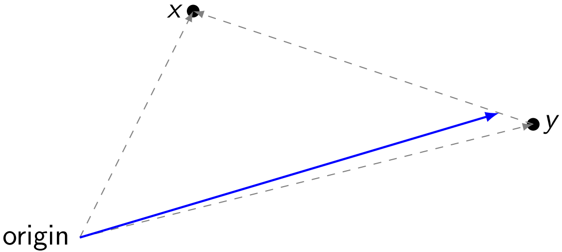


$\theta x + (1 - \theta) y$ with $\theta = 0$



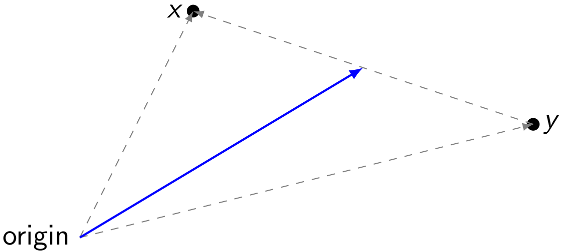
$$0x + (1 - 0)y = y$$

$\theta x + (1-\theta)y$ with $\theta = 0.1$



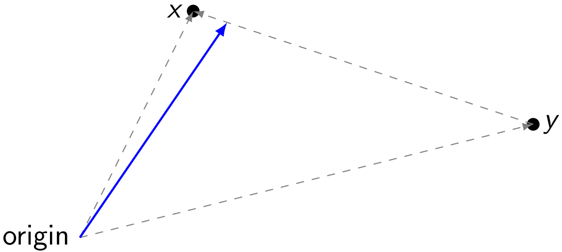
$$0.1x + (1-0.1)y = y + 0.1(x-y)$$

$\theta x + (1 - \theta) y$ with $\theta = 0.5$



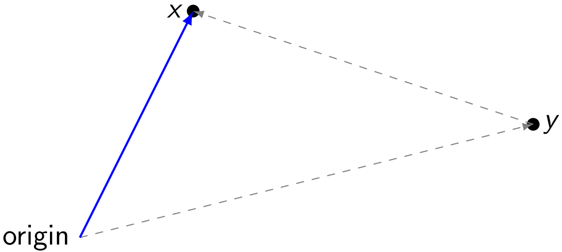
$$0.5x + (1 - 0.5)y = y + 0.5(x - y)$$

$\theta x + (1-\theta)y$ with $\theta = 0.9$



$$0.9x + (1-0.9)y = y + 0.9(x-y)$$

$\theta x + (1-\theta)y$ with $\theta = 1$



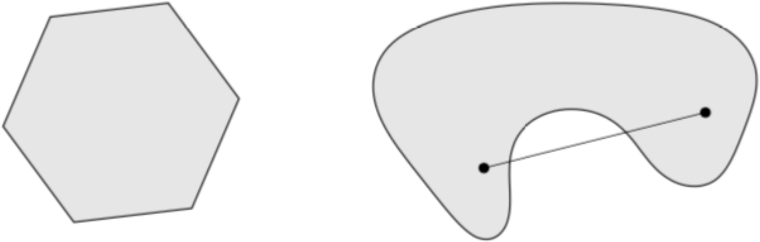
$$1x + (1-1)y = x$$

Convex sets

- a set $C \subseteq \mathbf{R}^n$ is **convex** if for all x, $y \in C$ and $\theta \in [0, 1]$,

$$\theta x + (1 - \theta) y \in C$$

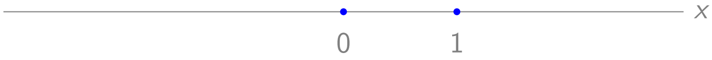
- C contains the line segment connecting any two points in C



Boyd and Vandenberghe (2004), Convex Optimization

Nonconvex subsets of $\mathbf{R}$

$\{0, 1\}$



$\{x \in \mathbf{R} \mid |x| \geq 1\}$

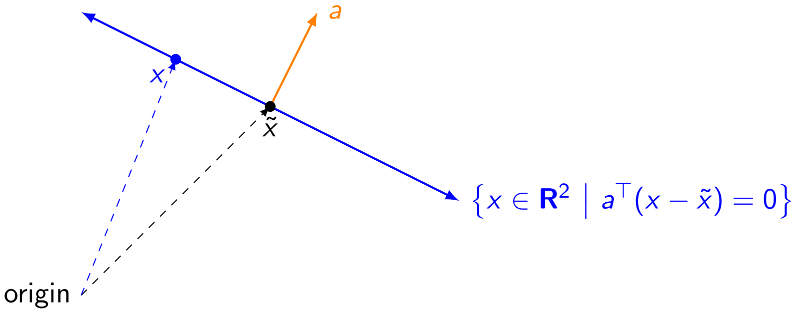
Hyperplanes

- any $b \in \mathbf{R}$ and nonzero $a \in \mathbf{R}^n$ define a hyperplane,

$$\left\{x \in \mathbf{R}^n \;\middle|\; a^\top x = b\right\}$$

- equivalent representation for any $\tilde{x}$ satisfying $a^\top \tilde{x} = b$:

$$\left\{x \in \mathbf{R}^n \;\middle|\; a^\top (x - \tilde{x}) = 0\right\}$$



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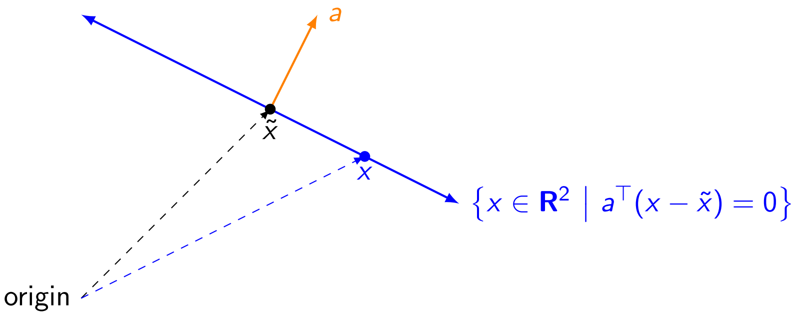
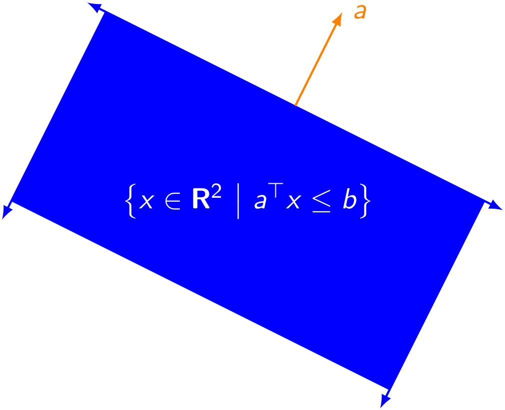


Halfspaces

any $a \neq 0$ and b (or $\tilde{x}$ with $a^\top \tilde{x} = b$) define a halfspace,

$$\left\{x \in \mathbf{R}^n \;\middle|\; a^\top x \leq b\right\} = \left\{x \in \mathbf{R}^n \;\middle|\; a^\top (x - \tilde{x}) \leq 0\right\}$$



Hyperplanes and halfspaces are convex

if $a^\top x \leq b$ and $a^\top y \leq b$, then for any $\theta \in [0, 1]$,

$$\begin{aligned} a^\top(\theta x + (1-\theta)y) &= \theta a^\top x + (1-\theta)a^\top y \\ &\leq \theta b + (1-\theta)b \\ &= b \end{aligned}$$

Intersections of convex sets are convex

- suppose sets $C_i \subseteq \mathbf{R}^n$ are convex for $i = 1, 2, \ldots$
- take any x, $y \in \bigcap_i C_i$
  (this just means that for all i, both x and y are in C_i)
- each C_i is convex, so for any $\theta \in [0, 1]$,

$$\theta x + (1 - \theta)y \in C_i$$

- since $\theta x + (1 - \theta)y \in C_i$ for all i, $\theta x + (1 - \theta)y \in \bigcap_i C_i$

- a **polyhedron** is a set

$$\left\{ x \in \mathbf{R}^n \;\middle|\; \begin{array}{l} a_i^\top x \le b_i \text{ for } i = 1, \ldots, m \\ c_j^\top x = d_j \text{ for } j = 1, \ldots, p \end{array} \right\}$$

  of solutions to finitely many linear inequalities and equations

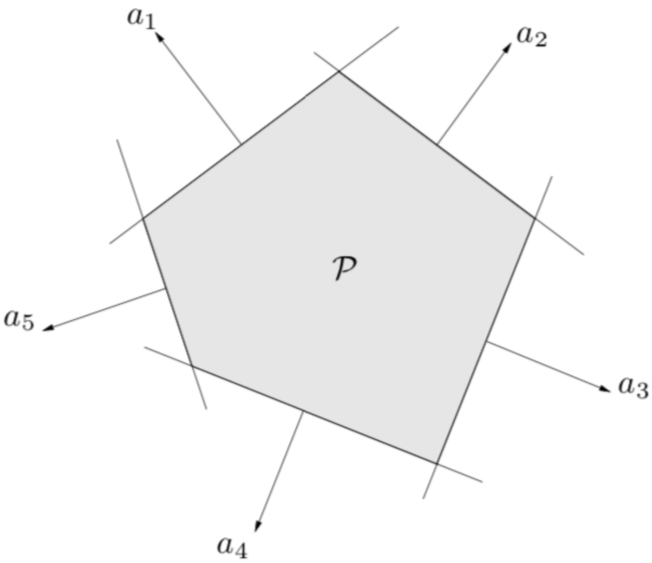
- a polyhedron can be written as

$$\left(\bigcap_{i=1}^{m} \left\{ x \in \mathbf{R}^n \;\middle|\; a_i^\top x \le b_i \right\} \right) \bigcap \left(\bigcap_{j=1}^{p} \left\{ x \in \mathbf{R}^n \;\middle|\; c_j^\top x = b_j \right\} \right),$$

  the intersection of m halfspaces and p hyperplanes

$\Longrightarrow$ polyhedra are convex

Polyhedra (continued)



Outline

Domain

- the **domain** of $f : \mathbf{R}^n \to \mathbf{R}$ is

$$\mathbf{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) \text{ is defined}\}$$

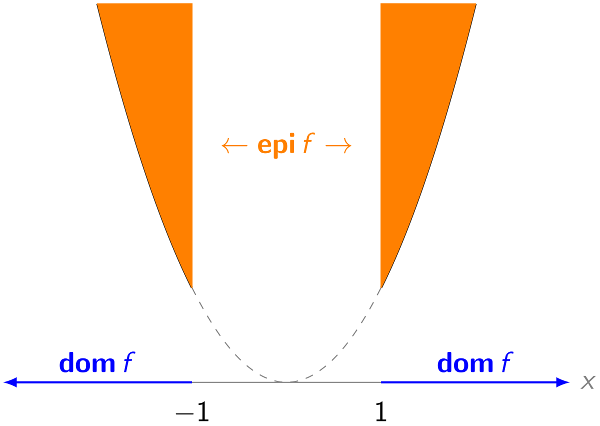
- example: for $\log : \mathbf{R} \to \mathbf{R}$, $\mathbf{dom}\log = \{x \in \mathbf{R} \mid x > 0\}$

Epigraph

- the **epigraph** of $f : \mathbf{R}^n \to \mathbf{R}$ is

$$\mathbf{epi}\, f = \left\{(x, y) \in \mathbf{R}^{n+1} \;\middle|\; x \in \mathbf{dom}\, f,\ y \geq f(x)\right\}$$

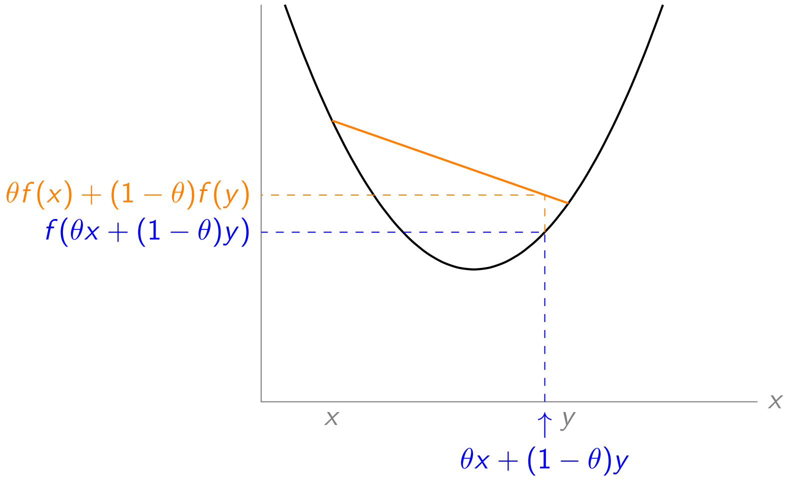
- example: $f(x) = x^2$, $\mathbf{dom}\, f = \{x \in \mathbf{R} \mid |x| \geq 1\}$



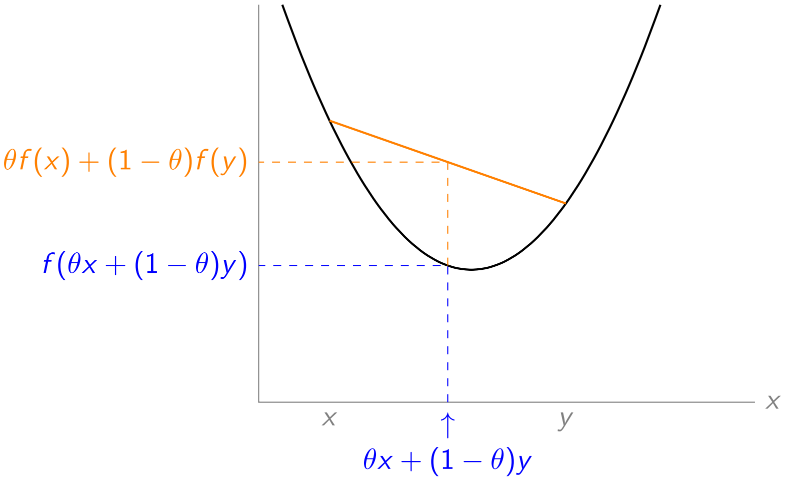
Convex functions

- $f : \mathbf{R}^n \to \mathbf{R}$ is **convex** if $\textbf{epi}\, f$ is convex
- equivalently,
  - ◇ $\textbf{dom}\, f$ is convex
  - ◇ for all $x, y \in \textbf{dom}\, f$ and $\theta \in [0, 1]$,

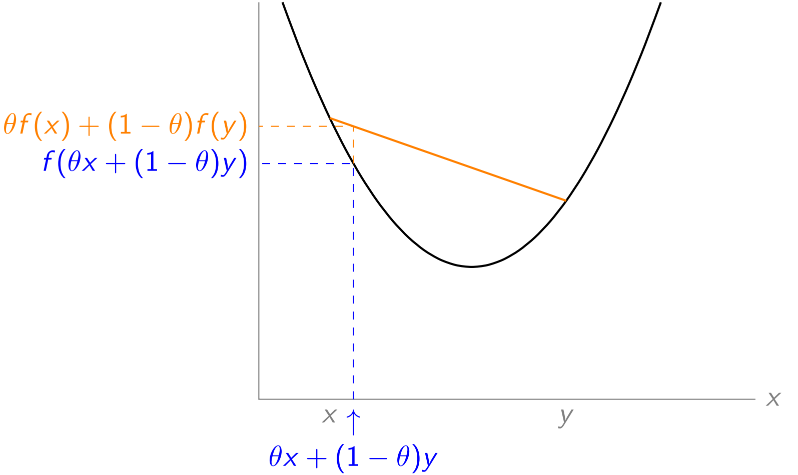
$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$


$\theta f(x) + (1-\theta)f(y)$
$f(\theta x + (1-\theta)y)$
x
x
y
$\theta x + (1-\theta)y$

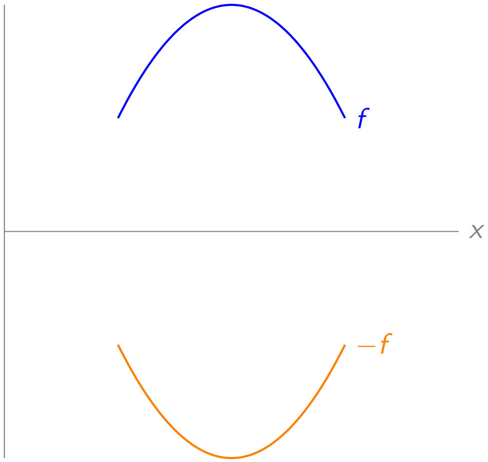
$\theta = 0.5$



$\theta = 0.9$



$f : \mathbf{R}^n \to \mathbf{R}$ is **concave** if $-f$ is convex



Affine functions are convex (and concave)

- $f : \mathbf{R}^n \to \mathbf{R}$ is **affine** if $f(x) = a^\top x + b$ for some a and b
- if f is affine, then f is convex (and concave):

$$\begin{aligned} f(\theta x + (1-\theta)y) &= a^\top(\theta x + (1-\theta)y) + b \\ &= \theta a^\top x + (1-\theta)a^\top y + b \\ &= \theta(a^\top x + b) + (1-\theta)(a^\top y + b) \\ &= \theta f(x) + (1-\theta)f(y) \end{aligned}$$

- conversely, any function that's convex and concave is affine

Outline

Convex sets

Convex functions

Composition rules

Example functions

Increasing and decreasing functions

- $f : \mathbf{R} \to \mathbf{R}$ is **nondecreasing** if

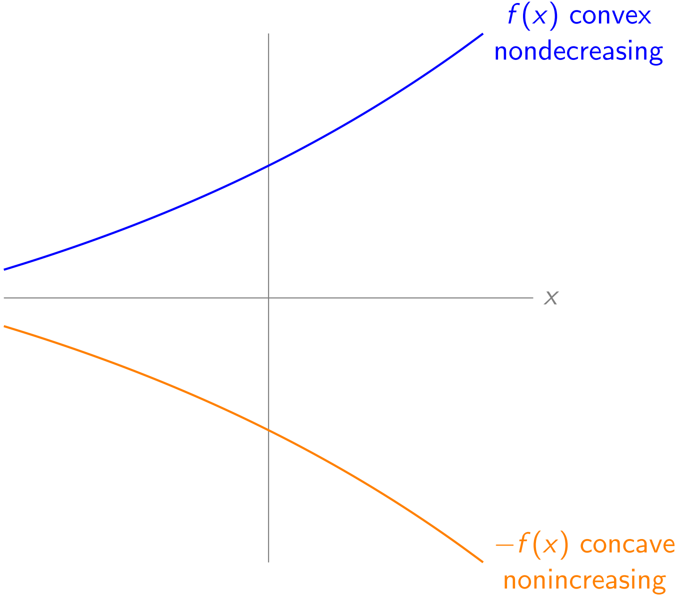
$$x \geq y \implies f(x) \geq f(y)$$

  (and **increasing** if $x > y \implies f(x) > f(y)$)
- similarly, f is **nonincreasing** if

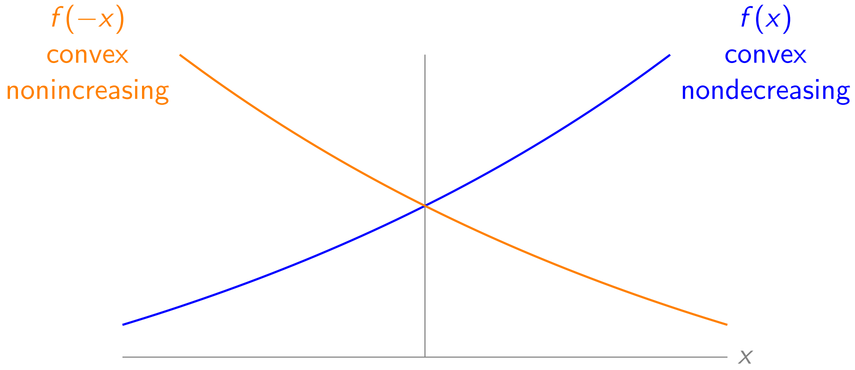
$$x \geq y \implies f(x) \leq f(y)$$

  (and **decreasing** if $x > y \implies f(x) < f(y)$)

$f(x)$ convex nondec. $\iff$ $-f(x)$ concave noninc.



$f(x)$ convex nondec. $\iff$ $f(-x)$ convex noninc.



The fundamental composition rule

- consider $h_1, \ldots, h_m : \mathbf{R}^n \to \mathbf{R}$ and convex $g : \mathbf{R}^m \to \mathbf{R}$
- define $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x) = g(h_1(x), \ldots, h_m(x))$
- f is convex if for each $i = 1, \ldots, m$,
  - h_i is affine, or
  - g is nondecreasing in argument i and h_i is convex, or
  - g is nonincreasing in argument i and h_i is concave
- less precisely but perhaps more memorably,
  - CVX(AFF) = CVX
  - CVXND(CVX) = CVX
  - CVXNI(CCV) = CVX

Composition rules for concave functions

- consider $h_1, \ldots, h_m : \mathbf{R}^n \to \mathbf{R}$ and concave $g : \mathbf{R}^m \to \mathbf{R}$
- define $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x) = g(h_1(x), \ldots, h_m(x))$
- f is concave if for each $i = 1, \ldots, m$,
  - h_i is affine, or
  - g is nondecreasing in argument i and h_i is concave, or
  - g is nonincreasing in argument i and h_i is convex

Useful special cases

- h_1, h_2 convex $\implies$ $h_1 + h_2$ convex
- h_1 convex, h_2 concave $\implies$ $h_1 - h_2$ convex
- h convex, $\alpha \geq 0$ $\implies$ αh convex
- h concave, $\alpha \geq 0$ $\implies$ αh concave
- h_i convex, $\alpha_i \geq 0$ $\implies$ $\alpha_1 h_1 + \cdots + \alpha_m h_m$ convex
- $h_1, \ldots, h_m$ convex $\implies$ $\max\{h_1, \ldots, h_m\}$ convex

Composition rules for monotonicity

- consider g, $h : \mathbf{R} \to \mathbf{R}$
- define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = g(h(x))$
- if g and h are nondecreasing, then f is nondecreasing:

$$x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y))$$

- if g and h are nonincreasing, then f is nondecreasing:

$$x \leq y \implies h(x) \geq h(y) \implies g(h(x)) \leq g(h(y))$$

- if g is NI and h is ND, then f is NI:

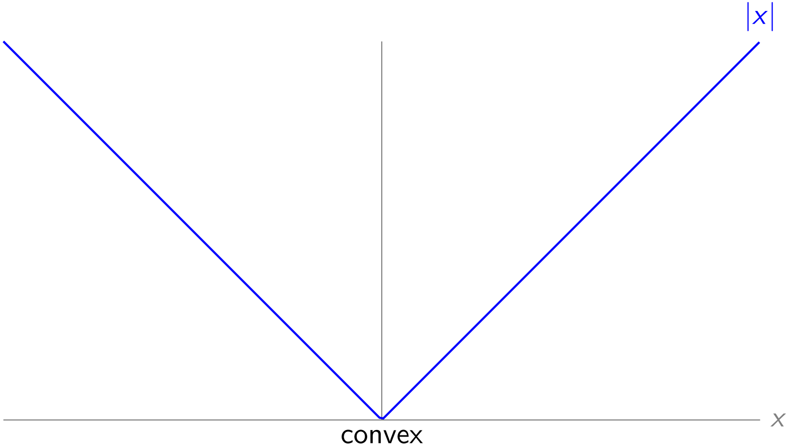
$$x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \geq g(h(y))$$

- if g is ND and h is NI, then f is NI:

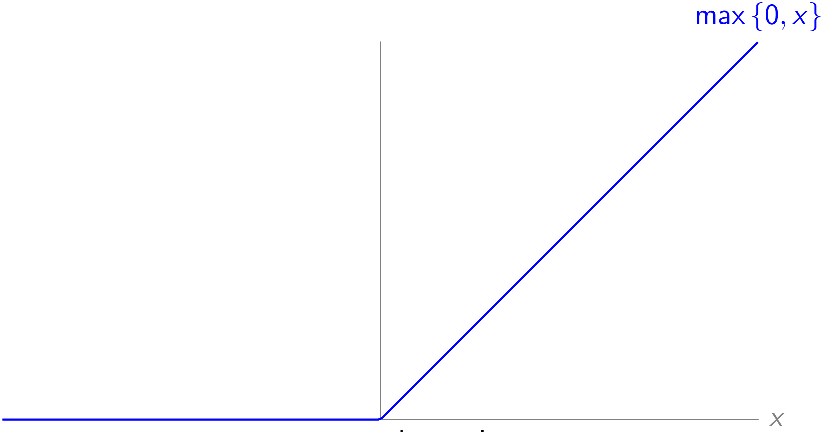
$$x \leq y \implies h(x) \geq h(y) \implies g(h(x)) \geq g(h(y))$$

Outline

$f(x) = |x|$ with $x \in \mathbf{R}$

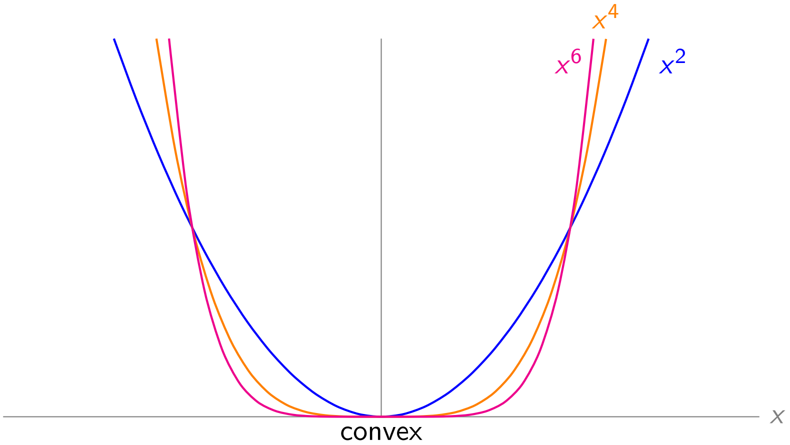


$f(x) = \max\{0, x\}$ with $x \in \mathbf{R}$

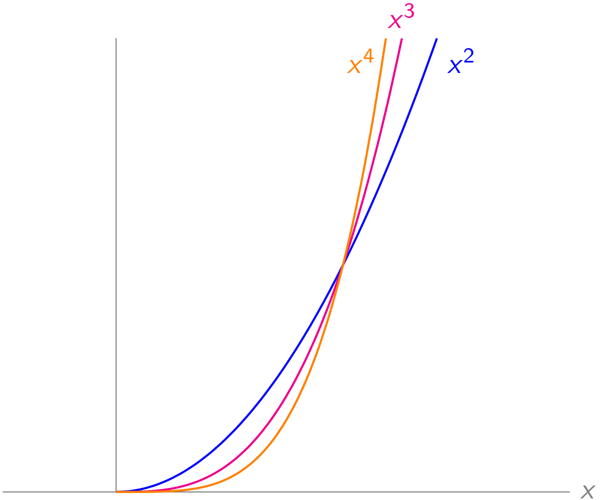


convex, nondecreasing

$f(x) = x^p$ with $x \in \mathbf{R}$ and even, positive p

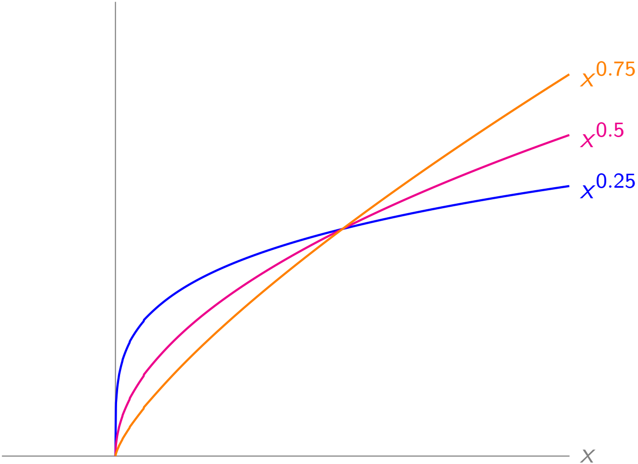


$f(x) = x^p$ with $x \geq 0$ and $p > 1$



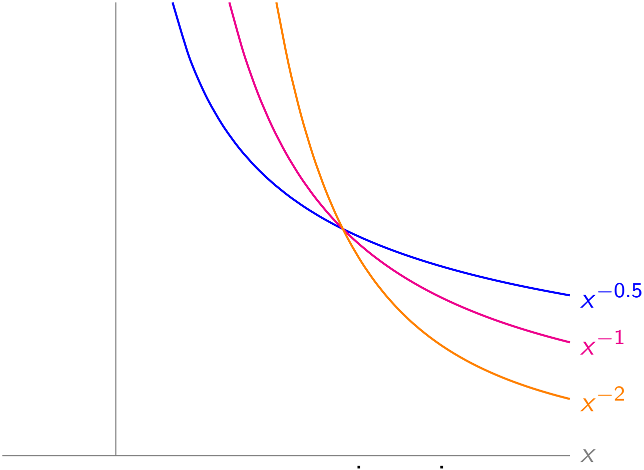
convex, nondecreasing

$f(x) = x^p$ with $x \geq 0$ and $p \in (0, 1)$



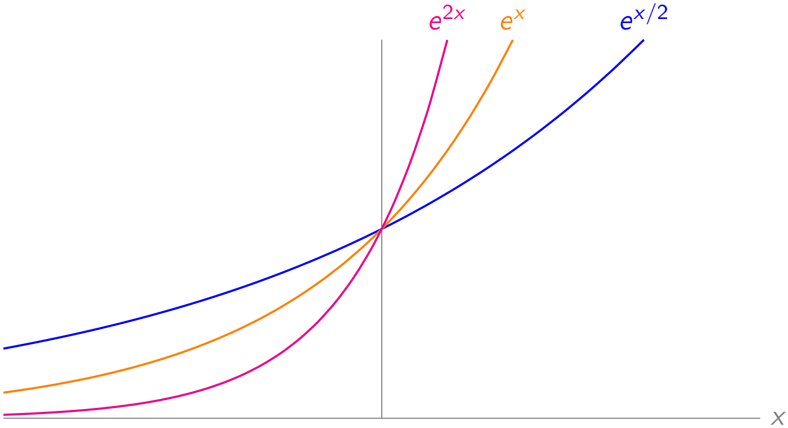
concave, nondecreasing

$f(x) = x^p$ with $x > 0$ and $p < 0$



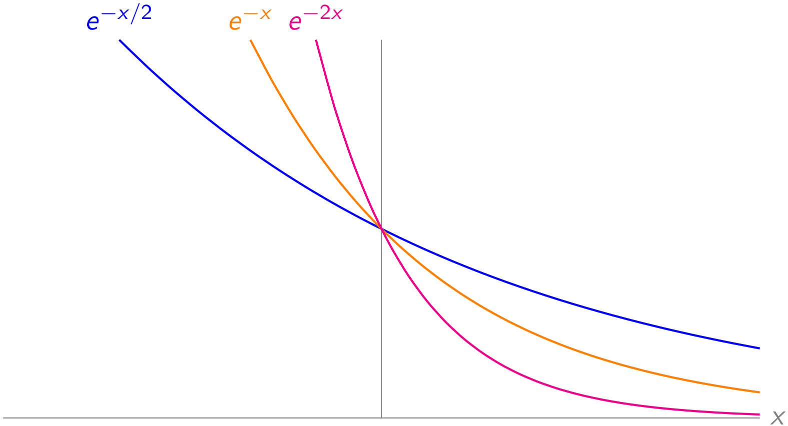
convex, nonincreasing

$f(x) = e^{\alpha x}$ with $x \in \mathbf{R}$, $\alpha \geq 0$



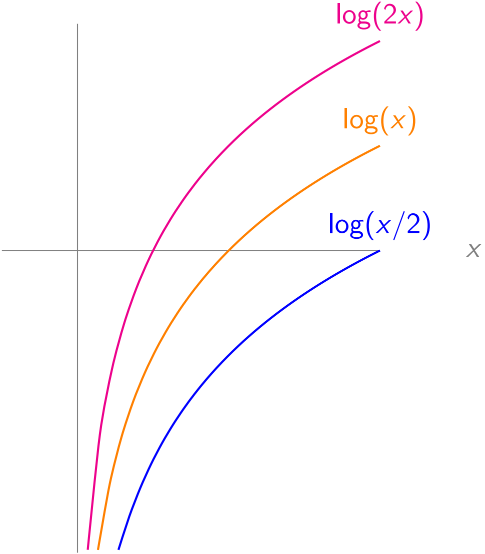
convex, nondecreasing

$f(x) = e^{\alpha x}$ with $x \in \mathbf{R}$, $\alpha < 0$



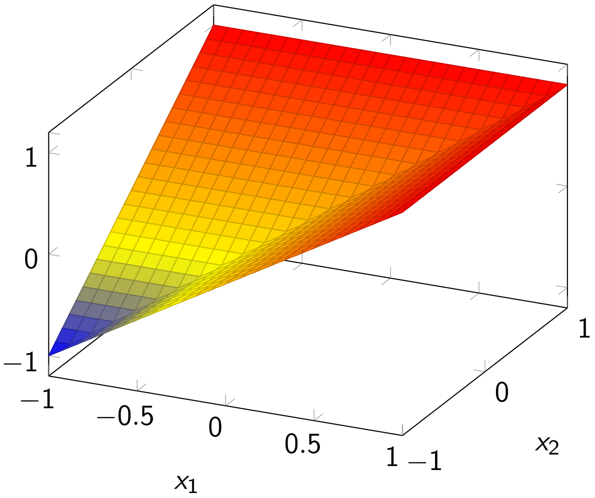
convex, nonincreasing

$f(x) = \log(\alpha x)$ with $x > 0$, $\alpha > 0$



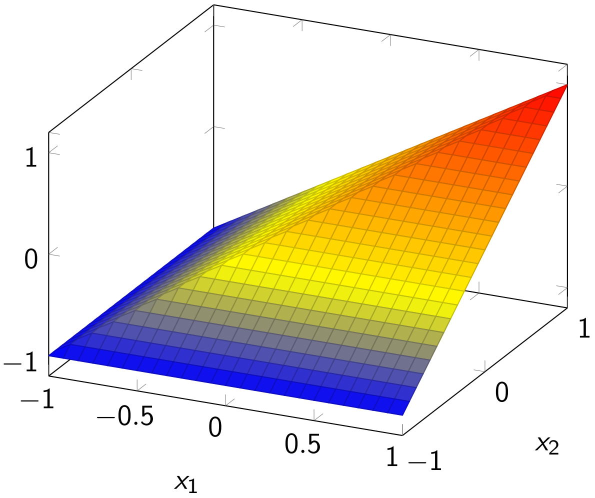
concave, nondecreasing

$f(x) = \max\{x_1, \ldots, x_n\}$ with $x \in \mathbf{R}^n$



convex, nondecreasing

$f(x) = \min\{x_1, \ldots, x_n\}$ with $x \in \mathbf{R}^n$



concave, nondecreasing

Norms

- $\| \ \| : \mathbf{R}^n \to \mathbf{R}$ is a **norm** if
  1. $\|x\| \geq 0$ for all $x \in \mathbf{R}^n$
  2. $\|x\| = 0 \iff x = 0$
  3. $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$
  4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbf{R}^n$
- all norms $\|x\|$
  - ◇ generalize the absolute value $|x|$ of $x \in \mathbf{R}$
  - ◇ provide different measures of the length of $x \in \mathbf{R}^n$ (or the distance $\|x - y\|$ between x and y)
  - ◇ are convex

Norm examples

- taxicab or ℓ_1 norm: $\|x\|_1 = |x_1| + \cdots + |x_n|$
- Euclidean or ℓ_2 norm: $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$
- Chebyshev or ℓ_∞ norm: $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$