Overview of optimization

Purdue ME 597, Distributed Energy Resources

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these slides draw on materials by Stephen Boyd at Stanford

Outline

Optimization problems

Optimization vocabulary

Tractable optimization problems

Our goal in studying optimization in ME 597

to become good **users** of convex optimization for DER applications

- optimization is a broad and deep field
- most optimization problems are intractable
- but **convex** problems are (usually) tractable
  - ◇ rich theory
  - ◇ efficient, reliable algorithms
  - ◇ convenient modeling software
  - ◇ often solved in subroutines for nonconvex problems
  - ◇ applications in engineering, science, economics, . . .
- we won't go deep, but you can (and should!) in other classes

Optimization problems

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- given $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$

Problem interpretation

- 'choose the best feasible n-vector'
- the **variable** $x = (x_1, \ldots, x_n)$ is the choice made
- the **objective** $f_0(x)$ quantifies 'how bad' x is
- x is **feasible** if
  - ◇ f_0, ..., f_m are all defined at x
    (for example, $\log : \mathbf{R} \to \mathbf{R}$ is defined only for $x > 0$)
  - ◇ x satisfies all the **constraints**: $f_1(x) \leq 0$, ..., $f_m(x) \leq 0$

Example: Solar photovoltaic array design

- choose solar array size (# panels or rated power) and orientation
- possible objectives:
  - initial cost (hardware, permitting, installation, ...)
  - electricity revenues or cost savings
  - greenhouse gas emission reductions
- possible constraints:
  - budget
  - usable rooftop or ground area
  - panel power output equations

Example: Electric vehicle charging

- choose charging powers at each time over a planning horizon
- possible objectives:
  - electricity costs
  - greenhouse gas emissions
  - peak electricity demand
- possible constraints:
  - battery energy and power capacities
  - battery dynamics
  - charging deadline

Equivalent problems

two problems are equivalent if

- a solution to the first readily yields a solution to the second
- and vice versa

Maximization and minimization

- suppose $g : \mathbf{R}^n \to \mathbf{R}$ quantifies 'how good' x is
- the maximization problem
  - choose $x \in \mathbf{R}^n$
  - to maximize $g(x)$
  - subject to $f_1(x) \le 0, \ldots, f_m(x) \le 0$

  is equivalent to the minimization problem
  - choose $x \in \mathbf{R}^n$
  - to minimize $-g(x)$
  - subject to $f_1(x) \le 0, \ldots, f_m(x) \le 0$

Constant objective terms

for any constant $a \in \mathbf{R}$, the problem

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x) + a$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

is equivalent to

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

Objective and constraint transformations

- suppose
  - $h : \mathbf{R} \to \mathbf{R}$ is increasing, meaning $y > z \implies h(y) > h(z)$
  - $g_1, \ldots, g_m : \mathbf{R} \to \mathbf{R}$ satisfy $g_i(y) \leq 0 \iff y \leq 0$
- then the problem
  - choose $x \in \mathbf{R}^n$
  - to minimize $f_0(x)$
  - subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

  is equivalent to
  - choose $x \in \mathbf{R}^n$
  - to minimize $h(f_0(x))$
  - subject to $g_1(f_1(x)) \leq 0, \ldots, g_m(f_m(x)) \leq 0$

Constraints with nonzero righthand sides

- for g, $h : \mathbf{R}^n \to \mathbf{R}$, the inequality constraint

$$g(x) \leq h(x)$$

  is equivalent to $f_1(x) \leq 0$ with $f_1(x) = g(x) - h(x)$

- similarly,

$$g(x) \geq h(x)$$

  is equivalent to $f_2(x) \leq 0$ with $f_2(x) = h(x) - g(x)$

Equality constraints

for $g,\ h : \mathbf{R}^n \to \mathbf{R}$, the equality constraint

$$g(x) = h(x)$$

is equivalent to the two inequality constraints

$$g(x) \leq h(x) \text{ and } g(x) \geq h(x),$$

which are equivalent to

$$f_1(x) \leq 0 \text{ and } f_2(x) \leq 0$$

with $f_1(x) = g(x) - h(x)$ and $f_2(x) = h(x) - g(x)$

Feasibility problems

- suppose we only want to
  - find any $x \in \mathbf{R}^n$
  - satisfying $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- this is equivalent to the optimization problem
  - choose $x \in \mathbf{R}^n$
  - to minimize 0
  - subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

Feasibility problems (example)

solving the system of nonlinear equations

$$g_1(x) = h_1(x),\ \ldots,\ g_m(x) = h_m(x)$$

is equivalent to solving the feasibility problem

- find $x \in \mathbf{R}^n$
- subject to $g_i(x) - h_i(x) \leq 0$, $h_i(x) - g(x) \leq 0$, $i = 1, \ldots, m$

Outline

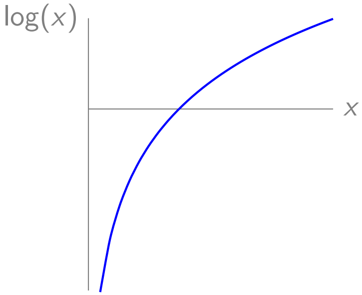
Infeasible and unbounded problems

a problem is

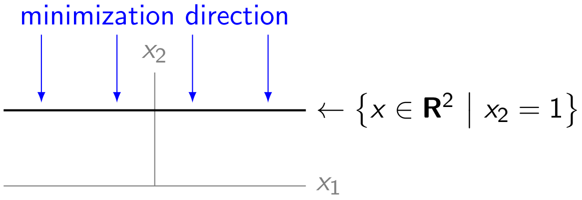
- **infeasible** if no feasible x exists
  example: minimize $x \in \mathbf{R}$ subject to $x \geq 2$, $x^2 \leq 1$
- **unbounded** if there is a sequence of feasible $x(k)$ such that

$$f_0(x(k)) \to -\infty \text{ as } k \to \infty$$

  example: minimize $\log(x)$ (take $x(1) = 1$, $x(k+1) = x(k)/2$)

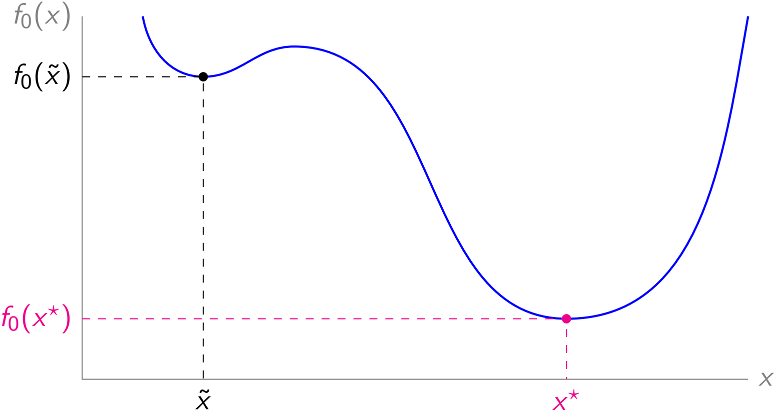


- an $x^\star \in \mathbf{R}^n$ is **optimal** (or an **optimizer**) if
  - $x^\star$ is feasible
  - $f_0(x^\star) \leq f_0(x)$ for all feasible x
- infeasible problems have no optimizers
- unbounded problems have no optimizers
- feasible, bounded problems can have multiple optimizers
  - example: choose $x \in \mathbf{R}^2$ to minimize x_2 subject to $x_2 = 1$



Local optimality

- an $\tilde{x}$ is **locally optimal** (or a **local optimizer**) if
  - ◇ $\tilde{x}$ is feasible
  - ◇ $f_0(\tilde{x}) \leq f_0(x)$ for all feasible x *in a neighborhood of* $\tilde{x}$
- an unlucky local optimizer $\tilde{x}$ might have $f_0(\tilde{x}) \gg f_0(x^\star)$



Outline

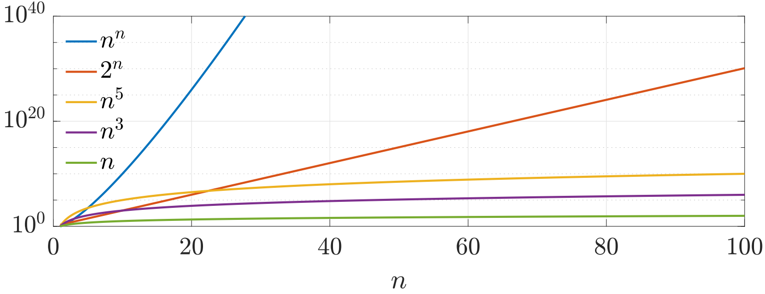
Optimization problems

Optimization vocabulary

Tractable optimization problems

Tractable optimization problems

- few optimization problems can be solved analytically
- but many can be solved numerically
- in general, *global* solve times grow exponentially in *n* and *m*
- often, *local* solve times grow only polynomially in *n* and *m*



Intractable example: The knapsack problem

- choose $x \in \mathbf{R}^n$
- to maximize $c^\top x$
- subject to $a^\top x \leq b$ and $x_1, \ldots, x_n \in \{0, 1\}$
- given $c \in \mathbf{R}^n$, $a \in \mathbf{R}^n$, $b \in \mathbf{R}$
- prove a polynomial-time algorithm? earn \$1 million

Local and global optimization

- a local optimizer $\tilde{x}$
  - can usually be computed efficiently
  - but might be far worse than a global $x^\star$ ($f_0(\tilde{x}) \gg f_0(x^\star)$)
- a global optimizer $x^\star$
  - gives the best feasible performance
  - but might be very slow to compute
- for **convex** problems, *all local optimizers are global optimizers* (more on convexity next lecture)

Least squares

- choose $x \in \mathbf{R}^n$
- to minimize $(Ax - b)^\top (Ax - b)$
- given $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $m \geq n$ (so A is tall)
- idea: no $x \in \mathbf{R}^n$ exactly satisfies all m equations in "$Ax = b$"
- so least squares finds an x with $Ax \approx b$
- analytical solution: $x^\star = (A^\top A)^{-1} A^\top b$ (`A\b` in Matlab)
  (assuming A has linearly independent columns)
- solve time is $\sim$proportional to $n^2 m$

Least squares solution

- for $f(x) = x^\top Px + q^\top x + r$ with $P = P^\top \in \mathbf{R}^{n\times n}$,

$$\nabla f(x) = 2Px + q$$

- least squares has $P = A^\top A = P^\top$, $q = -2A^\top b$:

$$\begin{aligned}(Ax-b)^\top(Ax-b) &= (x^\top A^\top - b^\top)(Ax-b)\\ &= x^\top A^\top Ax - x^\top A^\top b - b^\top Ax + b^\top b\\ &= x^\top A^\top Ax - 2b^\top Ax + b^\top b\end{aligned}$$

(recalling that $(CD)^\top = D^\top C^\top$ for matrices C and D)

- setting the gradient equal to zero gives

$$2A^\top Ax^\star - 2A^\top b = 0 \iff x^\star = (A^\top A)^{-1}A^\top b$$

provided $A^\top A$ is invertible ($\textbf{rank}\, A = n$)

One least squares interpretation: Model fitting

- b_i is observation i of a **target** we want to predict (e.g., a community's electricity demand)
- $A_{i1}, \ldots, A_{in}$ are observations i of n predictive **features** (e.g., outdoor temperature, hour, weekday, season, ...)
- $x_1, \ldots, x_n$ are **parameters** in a prediction model
- problem: choose x so that $x_1 A_{i1} + \cdots + x_n A_{in} \approx b_i$ for all i
- the least squares objective

$$(Ax - b)^\top (Ax - b) = \sum_{i=1}^{m} (x_1 A_{i1} + \cdots + x_n A_{in} - b_i)^2$$

penalizes errors between $x_1 A_{i1} + \cdots + x_n A_{in}$ and b_i for all i

Linear programming

- choose $x \in \mathbf{R}^n$
- to minimize $c^\top x$
- subject to $Ax \preceq b$
  (notation: for $y, z \in \mathbf{R}^n$, $y \preceq z$ means $y_1 \leq z_1, \ldots, y_n \leq z_n$)
- given $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$

- no analytical solution, but good algorithms
- solve time is ~proportional to $n^2 m$
- tricks can transform nonlinear problems into linear programs

Linear programming example: Chebyshev approximation

- x, A, b have same interpretations at least squares example (parameter vector, feature matrix, target vector)
- same goal: choose x so that $x_1 A_{i1} + \cdots + x_n A_{in} \approx b_i$ for all i
- instead of the least squares objective (sum of squared errors)

$$\sum_{i=1}^{m} (x_1 A_{i1} + \cdots + x_n A_{in} - b_i)^2,$$

  use the maximum absolute error

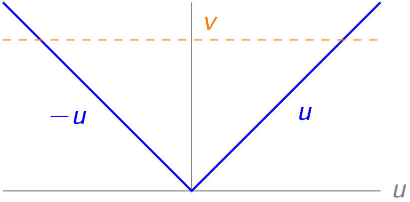
$$\max_{i=1,\ldots,m} |x_1 A_{i1} + \cdots + x_n A_{in} - b_i|$$

- this is not a linear program, but can be transformed into one

Chebyshev approximation as a linear program

- the Chebyshev approximation problem is to
  - choose $x \in \mathbf{R}^n$
  - to minimize $\max_{i=1,\ldots,m} |x_1 A_{i1} + \cdots + x_n A_{in} - b_i|$
- equivalently,
  - choose $(x, y) \in \mathbf{R}^{n+1}$
  - to minimize y
  - subject to $|x_1 A_{i1} + \cdots + x_n A_{in} - b_i| \leq y$, $i = 1, \ldots, m$
- still not a linear program, but closer

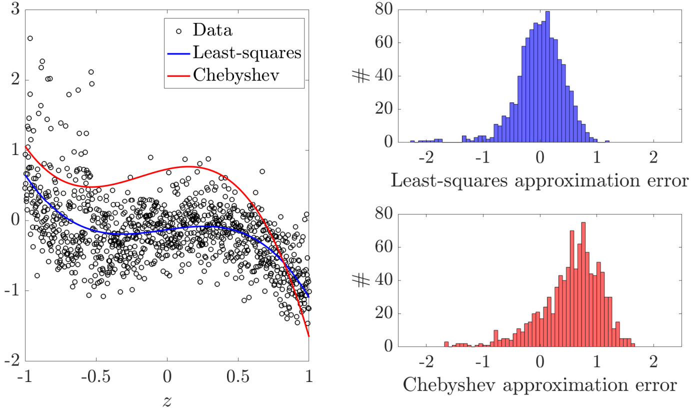
Chebyshev approximation as a linear program (continued)



- for any u, $v \in \mathbf{R}$, $|u| \leq v \iff u \leq v$ and $-u \leq v$
- so an equivalent problem to Chebyshev approximation is to
  - ◇ choose $(x, y) \in \mathbf{R}^{n+1}$
  - ◇ to minimize y
  - ◇ subject to
    $x_1 A_{i1} + \cdots + x_n A_{in} - b_i \leq y$, $i = 1, \ldots, m$
    $-(x_1 A_{i1} + \cdots + x_n A_{in} - b_i) \leq y$, $i = 1, \ldots, m$
- a linear program with $n + 1$ variables and $2m$ constraints

Model fitting example

- noisy data generated from unknown function of z: $b_i = f(z_i)$
- goal: approximate each b_i by cubic, $x_1 + x_2 z_i + x_3 z_i^2 + x_4 z_i^3$
- so $n = 4$ and $A_{ij} = z_i^{j-1}$



Convex optimization

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- given **convex** $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$

- no analytical solution, but good algorithms
- solve time is $\sim$proportional to $\max \left\{ n^3, n^2 m \right\}$
- includes least squares, linear programming, and much more

How to use convex optimization

- formulate your problem
- hopefully, recognize it as convex
- otherwise, reformulate or approximate it as convex
- code it in a convex modeling language
  (CVX, CVXPY, Convex.jl, CVXR, ...)
- tell modeling language to pass your problem to a solver
  (SeDuMi, SDPT3, Gurobi, MOSEK, GLPK, ...)
- check solution, tune formulation, repeat until satisfied

Coming soon

- convex sets and functions
- solving convex optimization problems
- DER optimization examples