Solving convex optimization problems

Purdue ME 597, Distributed Energy Resources

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these slides draw on materials by Stephen Boyd at Stanford

Outline

Disciplined convex programming in CVX

Examples

Optimization algorithms

Disciplined convex programming

- is a framework for describing convex optimization problems
- uses a library of functions with curvature, monotonicity tags
- imposes rules for compositions of functions
- is sufficient but not necessary for certifying convexity

Disciplined convex program structure

- (scalar) objective can be
  - minimize convex
  - maximize concave
  - omitted (for feasibility problems)
- constraints can be
  - convex <= concave
  - concave >= convex
  - affine == affine
  - omitted (for unconstrained problems)

⋆ affine functions are both convex and concave

CVX

- implements disciplined convex programming in Matlab
- transforms user-specified convex programs into standard form
- passes standard-form problems to solvers
- interprets solver status (solved, infeasible, unbounded, . . .)
- if solved, transforms solutions back to user-specified forms

CVX syntax

```
cvx_begin
  variable x(n,1)
  minimize( norm(x,Inf) )
  subject to
    A*x == b
cvx_end
```

- constants $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$ are defined above CVX scope
- within CVX scope, x is a variable
- after `cvx_end`, CVX populates
  - `cvx_status` with solver's exit status
  - x with solution (if `cvx_status` is `Solved`)

CVX syntax (continued)

- indentation doesn't matter
- '`subject to`' is unnecessary, but can improve readability
- equality constraints use ==, not = (assignment)
- CVX interprets inequalities like `x >= 0` elementwise
- CVX does not require an initial guess or function derivatives

Infeasible problems

if problem instance is infeasible, CVX populates

- `cvx_status` with `Infeasible`
- each element of x with `NaN`

Unbounded problems

- if problem instance is unbounded, CVX populates
  - ◇ `cvx_status` with `Unbounded`
  - ◇ x with a **direction** in which problem is unbounded
- the direction x is likely not feasible, but for any feasible $\tilde{x}$,
  - ◇ $\tilde{x} + \alpha x$ is feasible for all $\alpha \geq 0$
  - ◇ objective value of $\tilde{x} + \alpha x$ improves without bound as $\alpha \to \infty$
- to get a feasible $\tilde{x}$, omit objective and re-solve as feasibility problem

Some example functions

function	meaning	attributes
`max(x)`	$\max\{x_1, \ldots, x_n\}$	convex nondecreasing
`min(x)`	$\min\{x_1, \ldots, x_n\}$	concave nondecreasing
`pos(x)`	$\max\{0, x\}$	convex nondecreasing
`square_pos(x)`	$\max\{0, x\}^2$	convex nondecreasing
`inv_pos(x)`	$1/x$ (for $x > 0$)	convex nonincreasing
`sqrt(x)`	$\sqrt{x}$ (for $x \geq 0$)	concave nondecreasing
`norm(x,p)`	$\|x\|_p$	convex
`sum_square(x)`	$x_1^2 + \cdots + x_n^2$	convex

Quadratic forms

- for $P \in \mathbf{R}^{n \times n}$, $x^\top P x$ is a **quadratic form** in $x \in \mathbf{R}^n$
- can assume P is symmetric; if it's not, replace P by $(P + P^\top)/2$:

$$\begin{aligned} x^\top (P + P^\top) x/2 &= (x^\top P x + x^\top P^\top x)/2 \\ &= (x^\top P x + (x^\top P^\top x)^\top)/2 \\ &= (x^\top P x + x^\top P x)/2 \\ &= x^\top P x \end{aligned}$$

- in CVX, $x^\top P x$ is `quad_form(x,P)`

Convexity and quadratic forms

- a symmetric $P \in \mathbf{R}^{n\times n}$ is **positive semidefinite** ($P \succeq 0$) if

$$x^\top Px \geq 0 \text{ for all } x$$

$(\iff \mathbf{det}\, P \geq 0 \iff \lambda_i \geq 0$ for all eigenvalues λ_i of $P)$

- a symmetric $P \in \mathbf{R}^{n\times n}$ is **positive definite** ($P \succ 0$) if

$$x^\top Px > 0 \text{ for all } x \neq 0$$

$(\iff \mathbf{det}\, P > 0 \iff \lambda_i > 0$ for all eigenvalues λ_i of $P)$

- the quadratic form $x^\top Px$ is
  - ◇ convex if $P \succeq 0$
  - ◇ strictly convex (so has a **unique** global minimum) if $P \succ 0$

Quadratic forms in CVX

- `quad_form` and `sum_square` tend to be slow
- using `norm` instead can improve speed and accuracy
- for example, minimizing the least squares objective

$$\texttt{sum\_square(A*x - b)} = (Ax-b)^\top(Ax-b) = \|Ax-b\|_2^2$$

  can typically be done faster by minimizing

$$\texttt{norm(A*x - b)} = \|Ax-b\|_2 = \sqrt{\|Ax-b\|_2^2}$$

- these problems are equivalent since
  - ◇ if g is increasing, minimize $g(f(x)) \iff$ minimize $f(x)$
  - ◇ $\sqrt{\cdot}$ with nonnegative arguments is increasing
  - ◇ $\|\cdot\|_2^2$ is nonnegative

Quadratic forms in CVX (continued)

- another example: (convex) constraint $x^\top Px \le c$ with $x \in \mathbf{R}^n$
- if $P \succ 0$, it has a square root $R \in \mathbf{R}^{n\times n}$ with $R^\top R = P$
  (in Matlab, `R = chol(P)` computes an upper triangular R)
- since $\|y\|_2 = \sqrt{y^\top y}$,

$$\begin{aligned} & x^\top Px \le c \\ \iff & x^\top R^\top Rx \le c \\ \iff & \|Rx\|_2^2 \le c \\ \iff & \|Rx\|_2 \le \sqrt{c} \end{aligned}$$

- in CVX, `quad_form(x,P) <= c` usually works
- but `norm(chol(P)*x) <= sqrt(c)` is usually faster

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Least squares

- choose x to minimize $\|Ax - b\|_2^2$ given $A \in \mathbf{R}^{m \times n}$, b
- random problem instance:
  - $n = 500$, $m = 1000$
  - independent standard normal A and b
- computing the least squares solution

$$x^\star = (A^\top A)^{-1} A^\top b = \texttt{A\textbackslash b}$$

takes 0.0145 s on a 2.7 GHz processor

Least squares: CVX `sum_square` solution

```
cvx_begin
  variable x(n,1)
  minimize( sum_square(A*x - b) )
cvx_end
```

- solves in 2.32 s
- agrees with `A\b` to nine decimal places

Least squares: CVX `norm` solution

```
cvx_begin
  variable x(n,1)
  minimize( norm(A*x - b) )
cvx_end
```

- solves in 1.35 s (42% less than `sum_square`)
- also agrees with `A\b` to nine decimal places

Least squares: disciplined convex programming error

```
cvx_begin
  variable x(n,1)
  minimize( norm(A*x - b)^2 )
cvx_end
```

```
Disciplined convex programming error:
  Illegal operation:  {convex} .^ {2}
  (Consider POW_P, POW_POS, or POW_ABS instead.)
```

- square of norm matches no composition rule
  (a convex function of a convex function may not be convex)
- but CVX would allow `square_pos(norm(A*x - b))` since

$$\texttt{square\_pos(z)} = \max\{0, z\}^2$$

  is convex *and nondecreasing*

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Optimization algorithms

Why learn about optimization algorithms?

- tools like CVX require no knowledge of how solvers work
- but knowing a bit can help with debugging, interpreting results
- also, optimization algorithms can be clever and beautiful
- we'll just scratch the surface; other classes go much deeper

Smooth unconstrained convex optimization

- choose $x \in \mathbf{R}^n$
- to minimize $f(x)$
- given smooth convex $f : \mathbf{R}^n \to \mathbf{R}$

- optimality condition is $\nabla f(x^\star) = 0$ (n equations, n unknowns)
- for example, if $f(x) = x^\top P x + q^\top x + r$, then

$$\nabla f(x^\star) = 2Px^\star + q = 0$$

  is a system of linear equations that can be solved efficiently
  (if P is invertible, then $x^\star = -P^{-1}q/2$ is the unique solution)
- but general nonquadratic f require iterative methods

Iterative methods

iterative methods

- typically require an **initial guess** $x(0) \in \textbf{dom}\, f$
- produce a sequence of **iterates** $x(1), x(2), \ldots \in \textbf{dom}\, f$
- **converge** if $f(x(k)) \to f(x^\star)$ and $\nabla f(x(k)) \to 0$ as $k \to \infty$

Descent methods

- given initial guess $x(0) \in \mathbf{dom}\, f$, repeat:
  1. find a **descent direction** $d(k)$
  2. find a **step size** $\alpha(k)$
  3. update $x(k+1) = x(k) + \alpha(k)d(k)$
  4. increment k

  until a stopping condition (such as $\|\nabla f(x(k))\|$ small) holds
- descent direction and step size should satisfy
  - $x(k) + \alpha(k)d(k) \in \mathbf{dom}\, f$
  - $f(x(k) + \alpha(k)d(k)) < f(x(k))$

Finding a good step size $\alpha(k)$

- finding a good step size is called a **line search**
- if $\alpha(k)$ is too small, $f(x(k+1)) < f(x(k))$ but progress is slow
- if $\alpha(k)$ is too big, we risk $f(x(k+1)) > f(x(k))$
- one simple line search method:
  - while $f(x(k) + \alpha(k)d(k)) \geq f(x(k))$,
    - set $\alpha(k) \leftarrow \alpha(k)/2$
  - set $x(k+1) = x(k) + \alpha(k)d(k)$
  - set $\alpha(k+1) = 1.2\alpha(k)$

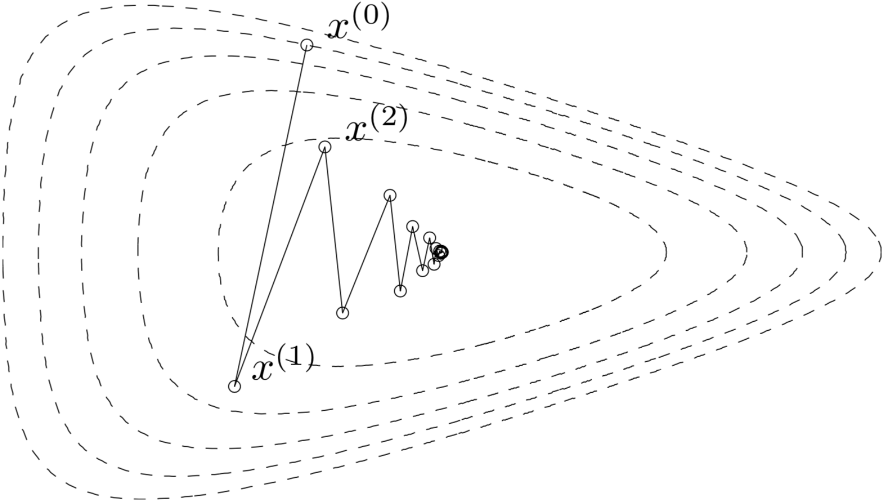
Gradient descent

- $-\nabla f(x)$ points in the direction of steepest descent of f at x
- so gradient descent uses descent direction

$$d(k) = -\nabla f(x(k))$$

- worst case: requires $\sim 1/\varepsilon$ iterations to get $f(x(k)) - f(x^\star) \leq \varepsilon$ (for example, $\sim 10^4$ iterations to get $f(x(k)) - f(x^\star) \leq 10^{-4}$)

Gradient descent illustration



Boyd and Vandenberghe (2004), Convex Optimization

Minimizing quadratic approximations

- Taylor's theorem: the quadratic approximation to f at $\tilde{x}$ is

$$\hat{f}(x) = f(\tilde{x}) + \nabla f(\tilde{x})^\top (x - \tilde{x}) + \frac{1}{2}(x - \tilde{x})^\top \nabla^2 f(\tilde{x})(x - \tilde{x})$$

- $\nabla^2 f(\tilde{x}) \in \mathbf{R}^{n \times n}$ is the second derivative (Hessian) matrix:

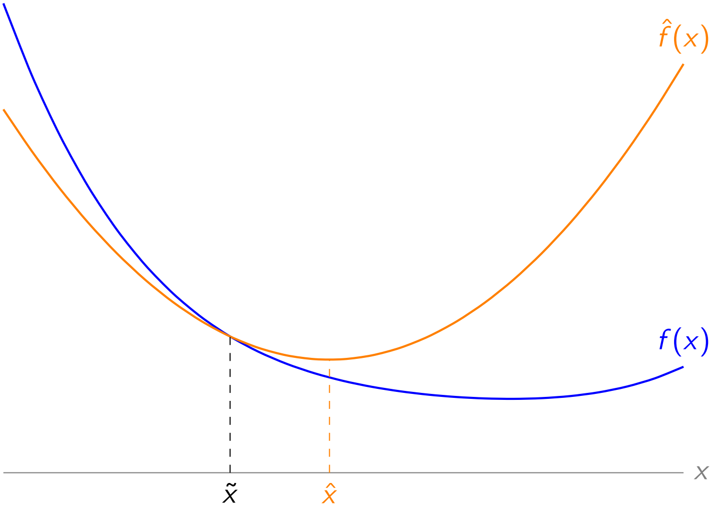
$$\nabla^2 f(\tilde{x})_{ij} = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\tilde{x}}$$

- some algebra shows that if the Hessian is invertible, then

$$\hat{x} = \tilde{x} - \nabla^2 f(\tilde{x})^{-1} \nabla f(\tilde{x})$$

minimizes $\hat{f}(x)$

Quadratic approximation illustration



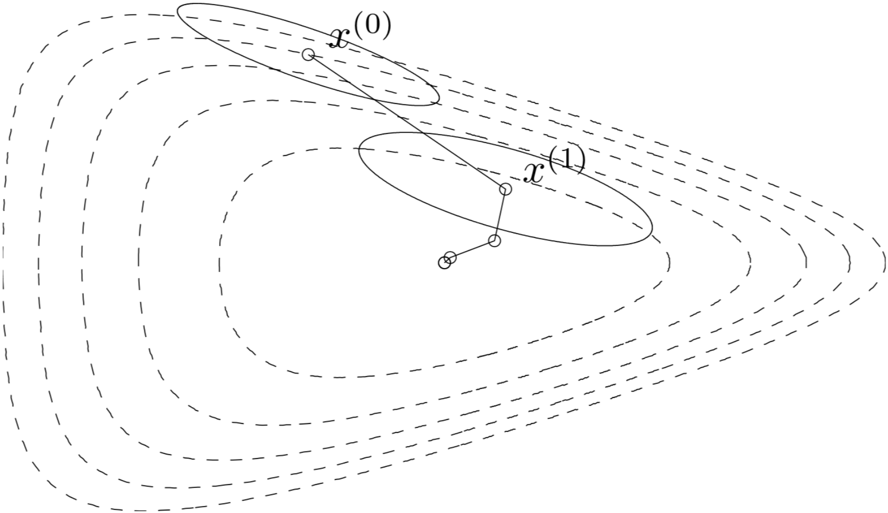
Newton's method

- Newton's method uses the descent direction

$$d(k) = -\nabla^2 f(x(k))^{-1} \nabla f(x(k))$$

  that minimizes the quadratic approximation to f at $x(k)$
- worst case: requires $\sim 1/\sqrt{\varepsilon}$ iterations to get $f(x(k)) - f(x^\star) \leq \varepsilon$ (for example, $\sim 10^2$ iterations to get $f(x(k)) - f(x^\star) \leq 10^{-4}$)

Newton's method illustration



Boyd and Vandenberghe (2004), Convex Optimization

Smooth constrained convex optimization

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- given smooth convex $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$

Logarithmic barrier

- equivalent problem: minimize $f_0(x) + \sum_{i=1}^{m} I_-(f_i(x))$, where

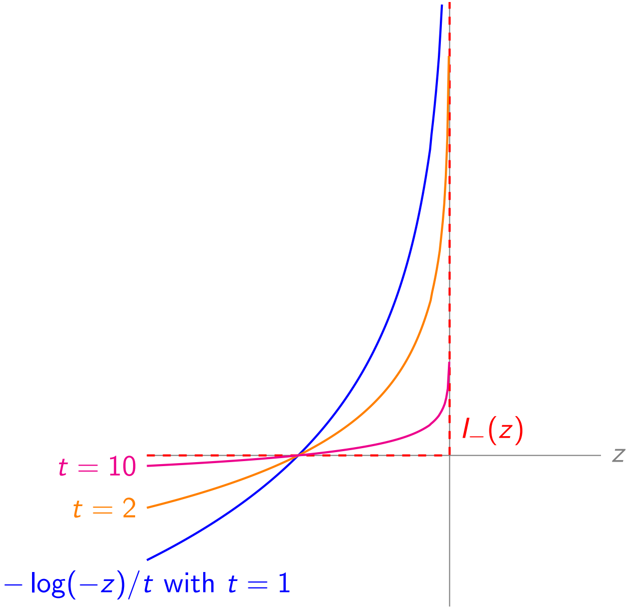
$$I_-(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ \infty & \text{otherwise} \end{cases}$$

  is the **indicator** function of $\{z \in \mathbf{R} \mid z \leq 0\}$
- idea: for a nondecreasing sequence of $t > 0$, minimize

$$f_0(x) - \frac{1}{t} \sum_{i=1}^{m} \log(-f_i(x))$$

- logarithmic barrier function $-\log(-z)/t$ approximates $I_-(z)$
- approximation improves as t increases

Logarithmic barrier approaches indicator as t increases



Barrier methods

- given $t(0) > 0$, $\gamma > 1$, initial guess $x(0) \in \textbf{dom}\, f_0$, repeat:
    1. set $x(k+1)$ by minimizing $f_0(x) - \frac{1}{t(k)} \sum_{i=1}^{m} \log(-f_i(x))$
    2. set $t(k+1) = \gamma t(k)$

  until a stopping condition (such as t large) holds
- step 1 typically uses Newton's method, initialized at $x(k)$
- trade-off: $\gamma \uparrow \implies$ outer iterations $\downarrow$ but Newton iterations $\uparrow$
- barrier methods converge at a rate similar to Newton's method

Interior-point methods

- are used by most solvers that CVX calls
- are conceptually similar to barrier methods
- do not need user-specified initial guesses or function derivatives
- have polynomial-time guarantees on worst-case complexity
- are often very fast in practice
- are typically faster for narrower problem classes:
    - ◇ linear programming (easiest)
    - ◇ quadratic programming
    - ◇ second-order cone programming
    - ◇ semidefinite programming (hardest)