Linear ordinary differential equations

Purdue ME 597, Distributed Energy Resources

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Outline

Notation and math reminders

First-order linear scalar ODEs

Battery example

Linear vector ODEs

Scalars

- a **scalar** is a number
- **R** is the set of **real** scalars
  (as opposed to integer, rational, imaginary, complex, ...)
- the notation $\alpha \in \mathbf{R}$ means α is a real scalar

Vectors

- a **vector** is an ordered list of numbers
- the **dimension** of a vector is the length of the list
- **column** vectors are vertical lists; **row** vectors are horizontal
- $\mathbf{R}^n$ is the set of real n-dimensional column vectors
- we write $a \in \mathbf{R}^n$ as

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{ or } (a_1, \ldots, a_n)$$

- $a_i \in \mathbf{R}$ is **element** i of $a \in \mathbf{R}^n$

Matrices

- a **matrix** is a rectangular array of numbers
- the **size** of a matrix is (# of rows) × (# of columns)
- $\mathbf{R}^{m \times n}$ is the set of real $m \times n$ matrices
- we write $A \in \mathbf{R}^{m \times n}$ as

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix}$$

- $A_{ij} \in \mathbf{R}$ is element i, j of A
- the **transpose** of $A \in \mathbf{R}^{m \times n}$ is

$$A^\top = \begin{bmatrix} A_{11} & \dots & A_{m1} \\ \vdots & & \vdots \\ A_{1n} & \dots & A_{mn} \end{bmatrix} \in \mathbf{R}^{n \times m}$$

Matrices generalize vectors generalize scalars

- a matrix $A \in \mathbf{R}^{n \times 1}$ with 1 column is a column vector
  (and a matrix $A \in \mathbf{R}^{1 \times n}$ with 1 row is a row vector)
- a 1-dimensional vector $a \in \mathbf{R}^1$ is a scalar
- for these reasons, we write $\mathbf{R}^{n \times 1}$ as $\mathbf{R}^n$ and $\mathbf{R}^1$ as $\mathbf{R}$

Scalar multiplication

for $\alpha \in \mathbf{R}$, $a \in \mathbf{R}^n$, and $A \in \mathbf{R}^{m \times n}$,

$$\alpha a = a\alpha = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix}$$

and

$$\alpha A = A\alpha = \begin{bmatrix} \alpha A_{11} & \dots & \alpha A_{1n} \\ \vdots & & \vdots \\ \alpha A_{m1} & \dots & \alpha A_{mn} \end{bmatrix}$$

Inner product

- for $a,\ b \in \mathbf{R}^n$,

$$a^\top b = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n b_n$$

- also called dot product, sometimes denoted $a \cdot b$ or $\langle a | b \rangle$
- example: $\mathbf{1}^\top a = a_1 + \cdots + a_n$, where $\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbf{R}^n$

Matrix-vector multiplication

- for $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^n$,

$$Ab = \begin{bmatrix} A_{11}b_1 + \cdots + A_{1n}b_n \\ \vdots \\ A_{m1}b_1 + \cdots + A_{mn}b_n \end{bmatrix}$$

$$=$$

$$=$$

- example: $Ib = b$, where $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbf{R}^{n \times n}$

Matrix-matrix multiplication

- for $A \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{n \times p}$, the i, j element of $AB \in \mathbf{R}^{m \times p}$ is

$$(AB)_{ij} =$$

- syntax AB only parses if $(\#\text{ columns of } A) = (\#\text{ rows of } B)$
- caution! $AB \neq BA$ in general
  - syntax $AB = BA$ only parses if A and B are both $n \times n$
  - even if A and B are both $n \times n$, $AB = BA$ only in special cases
- if $A \in \mathbf{R}^{n \times n}$ is invertible, then $A^{-1}A = AA^{-1} = I$

Matrix-valued functions of scalars

- $A : \mathbf{R} \to \mathbf{R}^{m \times n}$ means A is a function that
  - ◇ takes scalars as inputs
  - ◇ gives $m \times n$ matrices as outputs
- for $A : \mathbf{R} \to \mathbf{R}^{m \times n}$ and $t \in \mathbf{R}$, we write

$$A(t) = \begin{bmatrix} A_{11}(t) & \dots & A_{1n}(t) \\ \vdots & & \vdots \\ A_{m1}(t) & \dots & A_{mn}(t) \end{bmatrix}$$

- $A(t) \in \mathbf{R}^{m \times n}$ (an $m \times n$ matrix) is the value of A at t
- $A_{ij} : \mathbf{R} \to \mathbf{R}$ is element i, j of A
  (A_{ij} is a scalar-valued function of scalars)

Differentiating matrix-valued functions of scalars

- the derivative of $A : \mathbf{R} \to \mathbf{R}^{m \times n}$ is

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \begin{bmatrix} \frac{\mathrm{d}A_{11}(t)}{\mathrm{d}t} & \cdots & \frac{\mathrm{d}A_{1n}(t)}{\mathrm{d}t} \\ \vdots & & \vdots \\ \frac{\mathrm{d}A_{m1}(t)}{\mathrm{d}t} & \cdots & \frac{\mathrm{d}A_{mn}(t)}{\mathrm{d}t} \end{bmatrix}$$

- product rule: for $A : \mathbf{R} \to \mathbf{R}^{m \times n}$ and $b : \mathbf{R} \to \mathbf{R}^{n}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(A(t)b(t)) =$$

Integrating matrix-valued functions of scalars

- the integral of $A : \mathbf{R} \to \mathbf{R}^{m\times n}$ is

$$\int_{t_1}^{t_2} A(t)\mathrm{d}t = \begin{bmatrix} \int_{t_1}^{t_2} A_{11}(t)\mathrm{d}t & \dots & \int_{t_1}^{t_2} A_{1n}(t)\mathrm{d}t \\ \vdots & & \vdots \\ \int_{t_1}^{t_2} A_{m1}(t)\mathrm{d}t & \dots & \int_{t_1}^{t_2} A_{mn}(t)\mathrm{d}t \end{bmatrix}$$

- fundamental theorem of calculus: for $A : \mathbf{R} \to \mathbf{R}^{m\times n}$,

Block matrices

- the elements of a **block** matrix are matrices, e.g.

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

- submatrices B, C, D, and E must have consistent dimensions
  - ⋄ B and C must have the same # of rows
  - ⋄ D and E must have the same # of rows
  - ⋄ B and D must have the same # of columns
  - ⋄ C and E must have the same # of columns

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Scalar ordinary differential equations (ODEs)

- a scalar ODE
    - has a *scalar-valued function of scalars* as the variable
    - relates that function to its (ordinary) derivative(s)
- examples:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = e^{-t}x(t) - 3$$
$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \sin(x(t))$$
$$\frac{\mathrm{d}^3x(t)}{\mathrm{d}t^3} = t\frac{\mathrm{d}x(t)}{\mathrm{d}t} - x(t)$$

- solving these ODEs means finding the function $x : \mathbf{R} \to \mathbf{R}$

Categorizing ODEs

- the **order** of an ODE is the highest derivative it contains
- an *n*th-order ODE is **linear** if it can be written as

  $$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} = a_{n-1}(t)\frac{\mathrm{d}^{n-1}x(t)}{\mathrm{d}t^{n-1}} + \cdots + a_1(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_0(t)x(t) + b(t)$$

  for some functions $a_0, \ldots, a_{n-1}, b : \mathbf{R} \to \mathbf{R}$

ODE categorization examples

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = e^{-t}x(t) - 3$$

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \sin(x(t))$$

$$\frac{\mathrm{d}^3x(t)}{\mathrm{d}t^3} = t\frac{\mathrm{d}x(t)}{\mathrm{d}t} - x(t)$$

Solving first-order linear ODE initial value problems (IVPs)

- a general first-order linear scalar ODE IVP has the form

$$x(t^{\text{init}}) = x^{\text{init}}, \;\; \frac{\mathrm{d}x(t)}{\mathrm{d}t} = a(t)x(t) + b(t)$$

- multiplying the ODE by any positive $g : \mathbf{R} \to \mathbf{R}$ gives

- recall the product rule,

- so if $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = -a(t)g(t)$, then

What positive $g : \mathbf{R} \to \mathbf{R}$ satisfies $\frac{dg(t)}{dt} = -a(t)g(t)$?

- let $\int a(t)dt$ (itself a function of t) denote any antiderivative of a
    - ◇ example: if $a(t) = \cos(t)$, can use $\int a(t)dt = \sin(t)$
- guess: $g(t) = e^{-\int a(t)dt}$
    - ◇ this g is positive since $e^z > 0$ for any number $z \in \mathbf{R}$
- check:

$$\begin{aligned} \frac{dg(t)}{dt} &= \frac{d}{dt} e^{-\int a(t)dt} \\ &= \\ &= \\ &= -a(t)g(t) \end{aligned}$$

Solving first-order linear ODEs (continued)

- with $g(t) = e^{-\int a(t)\mathrm{d}t}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)g(t)) = g(t)b(t)$$

$$\Longrightarrow$$

$$\Longrightarrow$$

$$\Longrightarrow$$

- this is the solution x to our first-order linear ODE IVP

Summary: Solving first-order linear ODE IVPs

the solution to the first-order linear ODE IVP

$$x(t^{\text{init}}) = x^{\text{init}},\ \ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = a(t)x(t) + b(t)$$

is

$$x(t) = \frac{1}{g(t)}\left[g(t^{\text{init}})x^{\text{init}} + \int_{t^{\text{init}}}^{t} g(\tau)b(\tau)\mathrm{d}\tau\right]$$

where

$$g(t) = e^{-\int a(t)\mathrm{d}t}$$

Special cases with constant coefficients

- if a is constant, then $g(t) = e^{-ta}$ and

$$x(t) = e^{(t-t^{\text{init}})a}x^{\text{init}} + e^{ta}\int_{t^{\text{init}}}^{t} e^{-\tau a}b(\tau)\mathrm{d}\tau$$

- if b is also constant and $a \neq 0$, then

$$x(t) = e^{(t-t^{\text{init}})a}x^{\text{init}} + \frac{e^{(t-t^{\text{init}})a} - 1}{a}b$$

- if $a = 0$ and b is constant, then

$$x(t) = x^{\text{init}} + (t - t^{\text{init}})b,$$

as expected from the IVP $x(t^{\text{init}}) = x^{\text{init}},\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = b$

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A simple battery model

- a simple model of a battery is

- $x(t) \in \mathbf{R}$ (kWh) is the stored chemical potential energy
- $\tau > 0$ (h) is the self-dissipation time constant
- $p^{\text{chem}}(t)$ (kW) is the chemical charging power
  (or discharging if $p^{\text{chem}}(t) < 0$)

Solving a battery IVP with constant power

- the battery model is a first-order linear ODE with
  - $a = -1/\tau$ (constant)
  - $b(t) = p^{\text{chem}}(t)$
- so if $p^{\text{chem}}(t)$ is constant and $x(t^{\text{init}}) = x^{\text{init}}$, then

$$x(t) =$$

- as $t \to \infty$, $x(t)$ approaches a steady state $x^{\text{fin}} = \tau p^{\text{chem}}$:

$$x(t) = x^{\text{fin}} \implies \frac{\mathrm{d}x(t)}{\mathrm{d}t} =$$

The solution is a mixture of the initial and final states

- any **mixture** of quantities z_1 and z_2 can be written as

$$\lambda z_1 + (1-\lambda) z_2$$

  for some weight $\lambda \in [0,1]$
- since τ is positive, $e^{-(t-t^{\text{init}})/\tau} \in [0,1]$ for all $t \geq t^{\text{init}}$
- so (with constant p^{chem}) the battery IVP solution

$$x(t) = \underbrace{e^{-(t-t^{\text{init}})/\tau}}_{\lambda(t)} x^{\text{init}} + \underbrace{\left[1 - e^{-(t-t^{\text{init}})/\tau}\right]}_{1-\lambda(t)} x^{\text{fin}}$$

  is a mixture of x^{init} and x^{fin}, weighted by $\lambda(t) = e^{-(t-t^{\text{init}})/\tau}$

Convergence rate in terms of the time constant τ

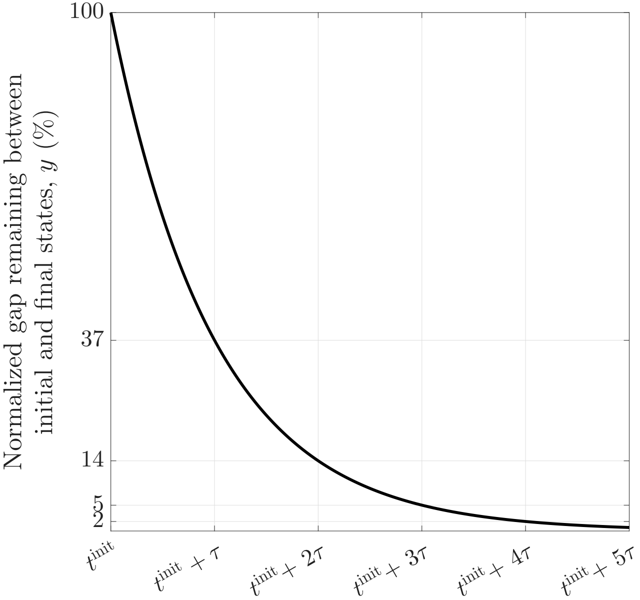
- define the normalized gap between $x(t)$ and x^{fin},

  $$y(t) = \frac{x^{\text{fin}} - x(t)}{x^{\text{fin}} - x^{\text{init}}},$$

  normalized by the initial gap $x^{\text{fin}} - x^{\text{init}}$
- a little algebra shows that $y(t) = e^{-(t - t^{\text{init}})/\tau}$
- so after n time constants, $100e^{-n}$% of the initial gap remains

t	t^{init}	$t^{\text{init}} + \tau$	$t^{\text{init}} + 2\tau$	$t^{\text{init}} + 3\tau$	$t^{\text{init}} + 4\tau$
$y(t)$	100%	37%	14%	5%	2%

Evolution of $y(t)$



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First-order linear vector ODEs

- a first-order linear **scalar** ODE has the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = a(t)x(t) + b(t)$$

  where $a(t) \in \mathbf{R}$, $b(t) \in \mathbf{R}$, and the variable is $x : \mathbf{R} \to \mathbf{R}$

- a first-order linear **vector** ODE has the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + b(t)$$

  where $A(t) \in \mathbf{R}^{n \times n}$, $b(t) \in \mathbf{R}^{n}$, and the variable is $x : \mathbf{R} \to \mathbf{R}^{n}$

- in terms of the matrix and vector elements,

Solving first-order linear vector ODE IVPs

- the first-order linear vector ODE IVP

$$x(t^{\mathrm{init}}) = x^{\mathrm{init}} \in \mathbf{R}^n,\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t)x(t) + b(t)$$

  has no analytical solution for general time-varying $A(t)$

- but for constant A, the IVP has solution

  where $e^M \in \mathbf{R}^{n\times n}$ is the **matrix exponential** of $M \in \mathbf{R}^{n\times n}$

- in Matlab, $e^M =$ `expm(M)`

The matrix exponential of any $M \in \mathbf{R}^{n\times n}$

- notation: $M^2 = MM$, $M^3 = MMM$, and so on
- definition:
$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$
- why define the matrix exponential? because for any $t \in \mathbf{R}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^{tM} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(I + tM + \frac{1}{2!}t^2M^2 + \frac{1}{3!}t^3M^3 + \dots\right)\\ &= \\ &= \\ &= Me^{tM}\end{aligned}$$

Properties of the matrix exponential of any $M \in \mathbf{R}^{n \times n}$

- $\frac{\mathrm{d}}{\mathrm{d}t} e^{tM} = Me^{tM} = e^{tM}M$ for any $t \in \mathbf{R}$
- $e^0 = I$ (where 0 and I are $n \times n$)
- $e^{(t_1+t_2)M} = e^{t_1 M} e^{t_2 M}$ for any $t_1,\ t_2 \in \mathbf{R}$
- e^M is always invertible and $(e^{tM})^{-1} = e^{-tM}$:

Properties of the matrix exponential (continued)

if M is invertible, then

$$\int_{t_1}^{t_2} e^{tM} \mathrm{d}t = M^{-1}\left(e^{t_2 M} - e^{t_1 M}\right) = \left(e^{t_2 M} - e^{t_1 M}\right) M^{-1}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tM} = M e^{tM}$$

$$\implies$$

$$\implies$$

$$\implies$$

Homework: Prove the linear vector ODE IVP solution

$$x(t^{\text{init}}) = x^{\text{init}} \in \mathbf{R}^n, \ \frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + b(t)$$

$$\implies x(t) = e^{(t-t^{\text{init}})A}x^{\text{init}} + e^{tA}\int_{t^{\text{init}}}^{t} e^{-\tau A}b(\tau)\mathrm{d}\tau$$

- follow the steps from the scalar ODE IVP proof
- use properties of the matrix exponential
- use the product rule: for $G : \mathbf{R} \to \mathbf{R}^{n\times n}$ and $x : \mathbf{R} \to \mathbf{R}^n$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(t)x(t)) = G(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{\mathrm{d}G(t)}{\mathrm{d}t}x(t)$$

Special case of invertible A, constant b

if A is invertible and b is constant, then

$$x(t) = e^{(t-t^{\text{init}})A} x^{\text{init}} + \left[e^{(t-t^{\text{init}})A} - I \right] A^{-1} b$$

since

$$\begin{aligned} e^{tA} \int_{t^{\text{init}}}^{t} e^{-\tau A} b \mathrm{d}\tau &= \\ &= \\ &= \\ &= \end{aligned}$$

Special case of noninvertible A, constant b

- if A and b are constant, then

$$x(t) = e^{(t-t^{\mathrm{init}})A}x^{\mathrm{init}} + e^{tA}\int_{t^{\mathrm{init}}}^{t} e^{-\tau A}\mathrm{d}\tau b$$

- how to compute $e^{tA}\int_{t^{\mathrm{init}}}^{t} e^{-\tau A}\mathrm{d}\tau b$ when A is noninvertible?
- compute $e^{(t-t^{\mathrm{init}})\bar{A}}$, where $\bar{A} = \begin{bmatrix} A & b \\ & \end{bmatrix} \in \mathbf{R}^{n+1\times n+1}$
- the upper right $n \times 1$ block of $e^{(t-t^{\mathrm{init}})\bar{A}}$ is $e^{tA}\int_{t^{\mathrm{init}}}^{t} e^{-\tau A}\mathrm{d}\tau b$

Special case of noninvertible A, constant b (proof)

- define the constant dummy variable $y(t) = 1$ and

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{(t-t^{\text{init}})A} & e^{tA}\int_{t^{\text{init}}}^{t} e^{-\tau A}\mathrm{d}\tau b \\ & 1 \end{bmatrix} \begin{bmatrix} x^{\text{init}} \\ 1 \end{bmatrix}$$

- then

$$z(t^{\text{init}}) = \begin{bmatrix} x^{\text{init}} \\ 1 \end{bmatrix}, \; \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \begin{bmatrix} \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}y(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} A & b \\ & \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \bar{A}z(t)$$

- this linear ODE IVP has solution $z(t) = e^{(t-t^{\text{init}})\bar{A}} \begin{bmatrix} x^{\text{init}} \\ 1 \end{bmatrix}$
- it follows that $\begin{bmatrix} e^{(t-t^{\text{init}})A} & e^{tA}\int_{t^{\text{init}}}^{t} e^{-\tau A}\mathrm{d}\tau b \\ & 1 \end{bmatrix} = e^{(t-t^{\text{init}})\bar{A}}$