

Linear ordinary differential equations

Purdue ME 597, Distributed Energy Resources

Kevin J. Kircher

Outline

Notation and math reminders

First-order linear scalar ODEs

Battery example

Linear vector ODEs

Scalars

- a **scalar** is a number
- **R** is the set of **real** scalars
(as opposed to integer, rational, imaginary, complex, ...)
- the notation $\alpha \in \mathbf{R}$ means α is a real scalar

Vectors

- a **vector** is an ordered list of numbers
- the **dimension** of a vector is the length of the list
- **column** vectors are vertical lists; **row** vectors are horizontal
- \mathbf{R}^n is the set of real n -dimensional column vectors
- we write $a \in \mathbf{R}^n$ as

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{ or } (a_1, \dots, a_n)$$

- $a_j \in \mathbf{R}$ is **element** i of $a \in \mathbf{R}^n$

Matrices

- a **matrix** is a rectangular array of numbers
- the **size** of a matrix is (# of rows) \times (# of columns)
- $\mathbf{R}^{m \times n}$ is the set of real $m \times n$ matrices
- we write $A \in \mathbf{R}^{m \times n}$ as

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix}$$

- $A_{ij} \in \mathbf{R}$ is element i, j of A
- the **transpose** of $A \in \mathbf{R}^{m \times n}$ is

$$A^T = \begin{bmatrix} A_{11} & \dots & A_{m1} \\ \vdots & & \vdots \\ A_{1n} & \dots & A_{mn} \end{bmatrix} \in \mathbf{R}^{n \times m}$$

Matrices generalize vectors generalize scalars

- a matrix $A \in \mathbf{R}^{n \times 1}$ with 1 column is a column vector
(and a matrix $A \in \mathbf{R}^{1 \times n}$ with 1 row is a row vector)
- a 1-dimensional vector $a \in \mathbf{R}^1$ is a scalar
- for these reasons, we write $\mathbf{R}^{n \times 1}$ as \mathbf{R}^n and \mathbf{R}^1 as \mathbf{R}

Scalar multiplication

for $\alpha \in \mathbf{R}$, $a \in \mathbf{R}^n$, and $A \in \mathbf{R}^{m \times n}$,

$$\alpha a = a\alpha = \begin{bmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{bmatrix}$$

and

$$\alpha A = A\alpha = \begin{bmatrix} \alpha A_{11} & \dots & \alpha A_{1n} \\ \vdots & & \vdots \\ \alpha A_{m1} & \dots & \alpha A_{mn} \end{bmatrix}$$

Inner product

- for $a, b \in \mathbf{R}^n$,

$$a^\top b = [a_1 \quad \dots \quad a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \dots + a_n b_n$$

- also called dot product, sometimes denoted $a \cdot b$ or $\langle a|b \rangle$

- example: $\mathbf{1}^\top a = a_1 + \dots + a_n$, where $\mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbf{R}^n$

Matrix-vector multiplication

- for $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^n$,

$$\begin{aligned} Ab &= \begin{bmatrix} A_{11}b_1 + \cdots + A_{1n}b_n \\ \vdots \\ A_{m1}b_1 + \cdots + A_{mn}b_n \end{bmatrix} \\ &= \begin{bmatrix} [A_{11} \ \cdots \ A_{1n}] b \\ \vdots \\ [A_{m1} \ \cdots \ A_{mn}] b \end{bmatrix} \\ &= b_1 \begin{bmatrix} A_{11} \\ \vdots \\ A_{m1} \end{bmatrix} + \cdots + b_n \begin{bmatrix} A_{1n} \\ \vdots \\ A_{mn} \end{bmatrix} \end{aligned}$$

- example: $Ib = b$, where $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbf{R}^{n \times n}$

Matrix-matrix multiplication

- for $A \in \mathbf{R}^{m \times n}$ and $B \in \mathbf{R}^{n \times p}$, the i, j element of $AB \in \mathbf{R}^{m \times p}$ is

$$(AB)_{ij} = [A_{i1} \quad \dots \quad A_{in}] \begin{bmatrix} B_{j1} \\ \vdots \\ B_{jn} \end{bmatrix}$$

- syntax AB only parses if ($\#$ columns of A) = ($\#$ rows of B)
- caution! $AB \neq BA$ in general
 - ◇ syntax $AB = BA$ only parses if A and B are both $n \times n$
 - ◇ even if A and B are both $n \times n$, $AB = BA$ only in special cases
- if $A \in \mathbf{R}^{n \times n}$ is invertible, then $A^{-1}A = AA^{-1} = I$

Matrix-valued functions of scalars

- $A : \mathbf{R} \rightarrow \mathbf{R}^{m \times n}$ means A is a function that
 - ◊ takes scalars as inputs
 - ◊ gives $m \times n$ matrices as outputs
- for $A : \mathbf{R} \rightarrow \mathbf{R}^{m \times n}$ and $t \in \mathbf{R}$, we write

$$A(t) = \begin{bmatrix} A_{11}(t) & \dots & A_{1n}(t) \\ \vdots & & \vdots \\ A_{m1}(t) & \dots & A_{mn}(t) \end{bmatrix}$$

- $A(t) \in \mathbf{R}^{m \times n}$ (an $m \times n$ matrix) is the value of A at t
- $A_{ij} : \mathbf{R} \rightarrow \mathbf{R}$ is element i, j of A
(A_{ij} is a scalar-valued function of scalars)

Differentiating matrix-valued functions of scalars

- the derivative of $A : \mathbf{R} \rightarrow \mathbf{R}^{m \times n}$ is

$$\frac{dA(t)}{dt} = \begin{bmatrix} \frac{dA_{11}(t)}{dt} & \cdots & \frac{dA_{1n}(t)}{dt} \\ \vdots & & \vdots \\ \frac{dA_{m1}(t)}{dt} & \cdots & \frac{dA_{mn}(t)}{dt} \end{bmatrix}$$

- product rule: for $A : \mathbf{R} \rightarrow \mathbf{R}^{m \times n}$ and $b : \mathbf{R} \rightarrow \mathbf{R}^n$,

$$\frac{d}{dt}(A(t)b(t)) = A(t)\frac{db(t)}{dt} + \frac{dA(t)}{dt}b(t)$$

Integrating matrix-valued functions of scalars

- the integral of $A : \mathbf{R} \rightarrow \mathbf{R}^{m \times n}$ is

$$\int_{t_1}^{t_2} A(t)dt = \begin{bmatrix} \int_{t_1}^{t_2} A_{11}(t)dt & \dots & \int_{t_1}^{t_2} A_{1n}(t)dt \\ \vdots & & \vdots \\ \int_{t_1}^{t_2} A_{m1}(t)dt & \dots & \int_{t_1}^{t_2} A_{mn}(t)dt \end{bmatrix}$$

- fundamental theorem of calculus: for $A : \mathbf{R} \rightarrow \mathbf{R}^{m \times n}$,

$$\int_{t_1}^{t_2} \frac{dA(t)}{dt} dt = A(t_2) - A(t_1)$$

Block matrices

- the elements of a **block** matrix are matrices, e.g.

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

- submatrices B , C , D , and E must have consistent dimensions
 - ◇ B and C must have the same # of rows
 - ◇ D and E must have the same # of rows
 - ◇ B and D must have the same # of columns
 - ◇ C and E must have the same # of columns

Outline

Notation and math reminders

First-order linear scalar ODEs

Battery example

Linear vector ODEs

Scalar ordinary differential equations (ODEs)

- a scalar ODE
 - ◊ has a *scalar-valued function of scalars* as the variable
 - ◊ relates that function to its (ordinary) derivative(s)
- examples:

$$\frac{dx(t)}{dt} = e^{-t}x(t) - 3$$

$$\frac{d^2x(t)}{dt^2} = \sin(x(t))$$

$$\frac{d^3x(t)}{dt^3} = t \frac{dx(t)}{dt} - x(t)$$

- solving these ODEs means finding the function $x : \mathbf{R} \rightarrow \mathbf{R}$

Categorizing ODEs

- the **order** of an ODE is the highest derivative it contains
- an n th-order ODE is **linear** if it can be written as

$$\frac{d^n x(t)}{dt^n} = a_{n-1}(t) \frac{d^{n-1} x(t)}{dt^{n-1}} + \dots + a_1(t) \frac{dx(t)}{dt} + a_0(t)x(t) + b(t)$$

for some functions $a_0, \dots, a_{n-1}, b : \mathbf{R} \rightarrow \mathbf{R}$

ODE categorization examples

$$\frac{dx(t)}{dt} = e^{-t}x(t) - 3$$

$$\frac{d^2x(t)}{dt^2} = \sin(x(t))$$

$$\frac{d^3x(t)}{dt^3} = t \frac{dx(t)}{dt} - x(t)$$

first-order, linear

second-order, nonlinear

third-order, linear

Solving first-order linear ODE initial value problems (IVPs)

- a general first-order linear scalar ODE IVP has the form

$$x(t^{\text{init}}) = x^{\text{init}}, \quad \frac{dx(t)}{dt} = a(t)x(t) + b(t)$$

- multiplying the ODE by any positive $g : \mathbf{R} \rightarrow \mathbf{R}$ gives

$$g(t) \frac{dx(t)}{dt} - x(t)a(t)g(t) = g(t)b(t)$$

- recall the product rule,

$$\frac{d}{dt}(x(t)g(t)) = \frac{dx(t)}{dt}g(t) + x(t)\frac{dg(t)}{dt}$$

- so if $\frac{dg(t)}{dt} = -a(t)g(t)$, then

$$\frac{d}{dt}(x(t)g(t)) = g(t)b(t)$$

What positive $g : \mathbf{R} \rightarrow \mathbf{R}$ satisfies $\frac{dg(t)}{dt} = -a(t)g(t)$?

- let $\int a(t)dt$ (itself a function of t) denote any antiderivative of a
 - ◊ example: if $a(t) = \cos(t)$, can use $\int a(t)dt = \sin(t)$
- guess: $g(t) = e^{-\int a(t)dt}$
 - ◊ this g is positive since $e^z > 0$ for any number $z \in \mathbf{R}$
- check:

$$\begin{aligned}\frac{dg(t)}{dt} &= \frac{d}{dt} e^{-\int a(t)dt} \\ &= e^{-\int a(t)dt} \frac{d}{dt} \left(-\int a(t)dt \right) \\ &= e^{-\int a(t)dt} (-a(t)) \\ &= -a(t)g(t)\end{aligned}$$

Solving first-order linear ODEs (continued)

- with $g(t) = e^{-\int a(t)dt}$, we have

$$\begin{aligned} & \frac{d}{dt}(x(t)g(t)) = g(t)b(t) \\ \implies & \int_{t^{\text{init}}}^t \frac{d}{d\tau}(x(\tau)g(\tau))d\tau = \int_{t^{\text{init}}}^t g(\tau)b(\tau)d\tau \\ \implies & x(t)g(t) - x(t^{\text{init}})g(t^{\text{init}}) = \int_{t^{\text{init}}}^t g(\tau)b(\tau)d\tau \\ \implies & x(t) = \frac{1}{g(t)} \left[g(t^{\text{init}})x^{\text{init}} + \int_{t^{\text{init}}}^t g(\tau)b(\tau)d\tau \right] \end{aligned}$$

- this is the solution x to our first-order linear ODE IVP

Summary: Solving first-order linear ODE IVPs

the solution to the first-order linear ODE IVP

$$x(t^{\text{init}}) = x^{\text{init}}, \quad \frac{dx(t)}{dt} = a(t)x(t) + b(t)$$

is

$$x(t) = \frac{1}{g(t)} \left[g(t^{\text{init}})x^{\text{init}} + \int_{t^{\text{init}}}^t g(\tau)b(\tau)d\tau \right]$$

where

$$g(t) = e^{-\int a(t)dt}$$

Special cases with constant coefficients

- if a is constant, then $g(t) = e^{-ta}$ and

$$x(t) = e^{(t-t^{\text{init}})a} x^{\text{init}} + e^{ta} \int_{t^{\text{init}}}^t e^{-\tau a} b(\tau) d\tau$$

- if b is also constant and $a \neq 0$, then

$$x(t) = e^{(t-t^{\text{init}})a} x^{\text{init}} + \frac{e^{(t-t^{\text{init}})a} - 1}{a} b$$

- if $a = 0$ and b is constant, then

$$x(t) = x^{\text{init}} + (t - t^{\text{init}})b,$$

as expected from the IVP $x(t^{\text{init}}) = x^{\text{init}}$, $\frac{dx(t)}{dt} = b$

Outline

Notation and math reminders

First-order linear scalar ODEs

Battery example

Linear vector ODEs

A simple battery model

- a simple model of a battery is

$$\frac{dx(t)}{dt} = -\frac{x(t)}{\tau} + p^{\text{chem}}(t)$$

- $x(t) \in \mathbf{R}$ (kWh) is the stored chemical potential energy
- $\tau > 0$ (h) is the self-dissipation time constant
- $p^{\text{chem}}(t)$ (kW) is the chemical charging power (or discharging if $p^{\text{chem}}(t) < 0$)

Solving a battery IVP with constant power

- the battery model is a first-order linear ODE with
 - ◊ $a = -1/\tau$ (constant)
 - ◊ $b(t) = p^{\text{chem}}(t)$
- so if $p^{\text{chem}}(t)$ is constant and $x(t^{\text{init}}) = x^{\text{init}}$, then

$$x(t) = e^{-(t-t^{\text{init}})/\tau} x^{\text{init}} + \left[1 - e^{-(t-t^{\text{init}})/\tau}\right] \tau p^{\text{chem}}$$

- as $t \rightarrow \infty$, $x(t)$ approaches a steady state $x^{\text{fin}} = \tau p^{\text{chem}}$:

$$x(t) = x^{\text{fin}} \implies \frac{dx(t)}{dt} = -\frac{\tau p^{\text{chem}}}{\tau} + p^{\text{chem}} = 0$$

The solution is a mixture of the initial and final states

- any **mixture** of quantities z_1 and z_2 can be written as

$$\lambda z_1 + (1 - \lambda) z_2$$

for some weight $\lambda \in [0, 1]$

- since τ is positive, $e^{-(t-t^{\text{init}})/\tau} \in [0, 1]$ for all $t \geq t^{\text{init}}$
- so (with constant p^{chem}) the battery IVP solution

$$x(t) = \underbrace{e^{-(t-t^{\text{init}})/\tau}}_{\lambda(t)} x^{\text{init}} + \underbrace{\left[1 - e^{-(t-t^{\text{init}})/\tau}\right]}_{1-\lambda(t)} x^{\text{fin}}$$

is a mixture of x^{init} and x^{fin} , weighted by $\lambda(t) = e^{-(t-t^{\text{init}})/\tau}$

Convergence rate in terms of the time constant τ

- define the normalized gap between $x(t)$ and x^{fin} ,

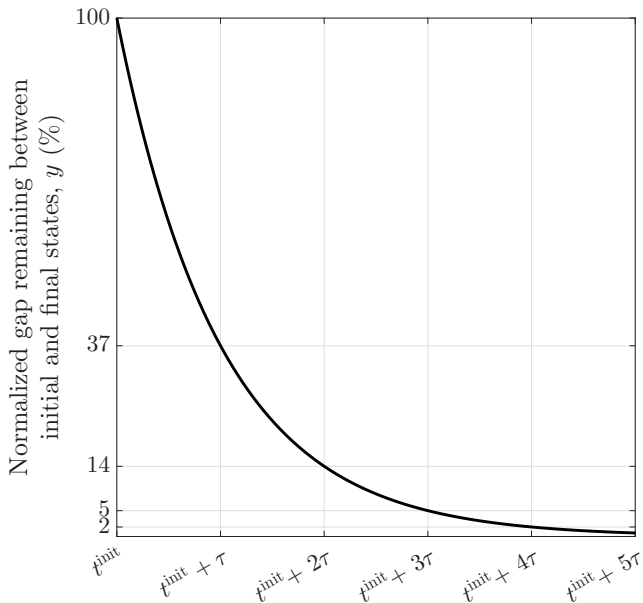
$$y(t) = \frac{x^{\text{fin}} - x(t)}{x^{\text{fin}} - x^{\text{init}}},$$

normalized by the initial gap $x^{\text{fin}} - x^{\text{init}}$

- a little algebra shows that $y(t) = e^{-(t-t^{\text{init}})/\tau}$
- so after n time constants, $100e^{-n}\%$ of the initial gap remains

t	t^{init}	$t^{\text{init}} + \tau$	$t^{\text{init}} + 2\tau$	$t^{\text{init}} + 3\tau$	$t^{\text{init}} + 4\tau$
$y(t)$	100%	37%	14%	5%	2%

Evolution of $y(t)$



Outline

Notation and math reminders

First-order linear scalar ODEs

Battery example

Linear vector ODEs

First-order linear vector ODEs

- a first-order linear **scalar** ODE has the form

$$\frac{dx(t)}{dt} = a(t)x(t) + b(t)$$

where $a(t) \in \mathbf{R}$, $b(t) \in \mathbf{R}$, and the variable is $x : \mathbf{R} \rightarrow \mathbf{R}$

- a first-order linear **vector** ODE has the form

$$\frac{dx(t)}{dt} = A(t)x(t) + b(t)$$

where $A(t) \in \mathbf{R}^{n \times n}$, $b(t) \in \mathbf{R}^n$, and the variable is $x : \mathbf{R} \rightarrow \mathbf{R}^n$

- in terms of the matrix and vector elements,

$$\begin{bmatrix} dx_1(t)/dt \\ \vdots \\ dx_n(t)/dt \end{bmatrix} = \begin{bmatrix} A_{11}(t) & \dots & A_{1n}(t) \\ \vdots & & \vdots \\ A_{n1}(t) & \dots & A_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

Solving first-order linear vector ODE IVPs

- the first-order linear vector ODE IVP

$$x(t^{\text{init}}) = x^{\text{init}} \in \mathbf{R}^n, \quad \frac{dx(t)}{dt} = A(t)x(t) + b(t)$$

has no analytical solution for general time-varying $A(t)$

- but for constant A , the IVP has solution

$$x(t) = e^{(t-t^{\text{init}})A} x^{\text{init}} + e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} b(\tau) d\tau$$

where $e^M \in \mathbf{R}^{n \times n}$ is the **matrix exponential** of $M \in \mathbf{R}^{n \times n}$

- in Matlab, $e^M = \text{expm}(M)$

The matrix exponential of any $M \in \mathbf{R}^{n \times n}$

- notation: $M^2 = MM$, $M^3 = MMM$, and so on
- definition:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

- why define the matrix exponential? because for any $t \in \mathbf{R}$,

$$\begin{aligned}\frac{d}{dt}e^{tM} &= \frac{d}{dt} \left(I + tM + \frac{1}{2!}t^2M^2 + \frac{1}{3!}t^3M^3 + \dots \right) \\ &= M + tM^2 + \frac{1}{2!}t^2M^3 + \dots \\ &= M \left(I + tM + \frac{1}{2!}t^2M^2 + \dots \right) \\ &= Me^{tM}\end{aligned}$$

Properties of the matrix exponential of any $M \in \mathbf{R}^{n \times n}$

- $\frac{d}{dt}e^{tM} = Me^{tM} = e^{tM}M$ for any $t \in \mathbf{R}$
- $e^0 = I$ (where 0 and I are $n \times n$)
- $e^{(t_1+t_2)M} = e^{t_1M}e^{t_2M}$ for any $t_1, t_2 \in \mathbf{R}$
- e^M is always invertible and $(e^{tM})^{-1} = e^{-tM}$:

$$e^{tM}e^{-tM} = e^{(t-t)M} = e^{0M} = e^0 = I$$

Properties of the matrix exponential (continued)

if M is invertible, then

$$\int_{t_1}^{t_2} e^{tM} dt = M^{-1} (e^{t_2 M} - e^{t_1 M}) = (e^{t_2 M} - e^{t_1 M}) M^{-1}$$

since

$$\begin{aligned} \frac{d}{dt} e^{tM} &= M e^{tM} \\ \implies \int_{t_1}^{t_2} \frac{d}{dt} e^{tM} dt &= M \int_{t_1}^{t_2} e^{tM} dt \\ \implies e^{t_2 M} - e^{t_1 M} &= M \int_{t_1}^{t_2} e^{tM} dt \\ \implies M^{-1} (e^{t_2 M} - e^{t_1 M}) &= \int_{t_1}^{t_2} e^{tM} dt \end{aligned}$$

Homework: Prove the linear vector ODE IVP solution

$$x(t^{\text{init}}) = x^{\text{init}} \in \mathbf{R}^n, \quad \frac{dx(t)}{dt} = Ax(t) + b(t)$$
$$\implies x(t) = e^{(t-t^{\text{init}})A}x^{\text{init}} + e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} b(\tau) d\tau$$

- follow the steps from the scalar ODE IVP proof
- use properties of the matrix exponential
- use the product rule: for $G : \mathbf{R} \rightarrow \mathbf{R}^{n \times n}$ and $x : \mathbf{R} \rightarrow \mathbf{R}^n$,

$$\frac{d}{dt}(G(t)x(t)) = G(t)\frac{dx(t)}{dt} + \frac{dG(t)}{dt}x(t)$$

Special case of invertible A , constant b

if A is invertible and b is constant, then

$$x(t) = e^{(t-t^{\text{init}})A} x^{\text{init}} + \left[e^{(t-t^{\text{init}})A} - I \right] A^{-1} b$$

since

$$\begin{aligned} e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} b d\tau &= e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} d\tau b \\ &= e^{tA} \left[- \left(e^{-\tau A} - e^{-t^{\text{init}} A} \right) A^{-1} \right] b \\ &= e^{tA} \left(e^{-t^{\text{init}} A} - e^{-tA} \right) A^{-1} b \\ &= \left[e^{(t-t^{\text{init}})A} - I \right] A^{-1} b \end{aligned}$$

Special case of noninvertible A , constant b

- if A and b are constant, then

$$x(t) = e^{(t-t^{\text{init}})A}x^{\text{init}} + e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} d\tau b$$

- how to compute $e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} d\tau b$ when A is noninvertible?
- compute $e^{(t-t^{\text{init}})\bar{A}}$, where $\bar{A} = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+1 \times n+1}$
- the upper right $n \times 1$ block of $e^{(t-t^{\text{init}})\bar{A}}$ is $e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} d\tau b$

Special case of noninvertible A , constant b (proof)

- define the constant dummy variable $y(t) = 1$ and

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{(t-t^{\text{init}})A} & e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} d\tau b \\ & 1 \end{bmatrix} \begin{bmatrix} x^{\text{init}} \\ 1 \end{bmatrix}$$

- then

$$z(t^{\text{init}}) = \begin{bmatrix} x^{\text{init}} \\ 1 \end{bmatrix}, \quad \frac{dz(t)}{dt} = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \end{bmatrix} = \begin{bmatrix} A & b \\ & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \bar{A}z(t)$$

- this linear ODE IVP has solution $z(t) = e^{(t-t^{\text{init}})\bar{A}} \begin{bmatrix} x^{\text{init}} \\ 1 \end{bmatrix}$
- it follows that $\begin{bmatrix} e^{(t-t^{\text{init}})A} & e^{tA} \int_{t^{\text{init}}}^t e^{-\tau A} d\tau b \\ & 1 \end{bmatrix} = e^{(t-t^{\text{init}})\bar{A}}$