Overview of optimization

Purdue ME 597, Distributed Energy Resources

Kevin J. Kircher

these slides draw on materials by Stephen Boyd at Stanford

Outline

Optimization problems

Optimization vocabulary

Tractable optimization problems

Our goal in studying optimization in ME 597

to become good **users** of **convex** optimization for DER applications

- optimization is a broad and deep field
- most optimization problems are intractable
- but convex problems are (usually) tractable
  - ◇ rich theory
  - ◇ efficient, reliable algorithms
  - ◇ convenient modeling software
  - ◇ often solved in subroutines for nonconvex problems
  - ◇ applications in engineering, science, economics, . . .
- we won't go deep, but you can (and should!) in other classes

Optimization problems

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- given $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$

Problem interpretation

- 'choose the best feasible n-vector'
- the **variable** $x = (x_1, \ldots, x_n)$ is the choice made
- the **objective** $f_0(x)$ quantifies 'how bad' x is
- x is **feasible** if
  - ⋄ f_0, . . . , f_m are all defined at x
    (for example, $\log : \mathbf{R} \to \mathbf{R}$ is defined only for $x > 0$)
  - ⋄ x satisfies all the **constraints**: $f_1(x) \leq 0$, . . . , $f_m(x) \leq 0$

Example: Solar photovoltaic array design

- choose solar array size (# panels or rated power) and orientation
- possible objectives:
  - initial cost (hardware, permitting, installation, . . .)
  - electricity revenues or cost savings
  - pollution
- possible constraints:
  - budget
  - usable rooftop or ground area
  - panel power output equations

Example: Electric vehicle charging

- choose charging powers at each time over a planning horizon
- possible objectives:
  - electricity costs
  - pollution
  - peak electricity demand
- possible constraints:
  - battery energy and power capacities
  - battery dynamics
  - charging deadline

Equivalent problems

two problems are equivalent if

- a solution to the first readily yields a solution to the second
- and vice versa

Maximization and minimization

- suppose $g : \mathbf{R}^n \to \mathbf{R}$ quantifies 'how good' x is
- the maximization problem
  - choose $x \in \mathbf{R}^n$
  - to maximize $g(x)$
  - subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

  is equivalent to the minimization problem
  - choose $x \in \mathbf{R}^n$
  - to minimize $-g(x)$
  - subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

Constant objective terms

for any constant $a \in \mathbf{R}$, the problem

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x) + a$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

is equivalent to

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

Objective and constraint transformations

- suppose
  - $h : \mathbf{R} \to \mathbf{R}$ is increasing, meaning $y > z \implies h(y) > h(z)$
  - $g_1, \ldots, g_m : \mathbf{R} \to \mathbf{R}$ satisfy $g_i(y) \leq 0 \iff y \leq 0$
- then the problem
  - choose $x \in \mathbf{R}^n$
  - to minimize $f_0(x)$
  - subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

  is equivalent to
  - choose $x \in \mathbf{R}^n$
  - to minimize $h(f_0(x))$
  - subject to $g_1(f_1(x)) \leq 0, \ldots, g_m(f_m(x)) \leq 0$

Constraints with nonzero righthand sides

- for g, $h : \mathbf{R}^n \to \mathbf{R}$, the inequality constraint

$$g(x) \leq h(x)$$

  is equivalent to $f_1(x) \leq 0$ with $f_1(x) = g(x) - h(x)$

- similarly,

$$g(x) \geq h(x)$$

  is equivalent to $f_2(x) \leq 0$ with $f_2(x) = h(x) - g(x)$

Equality constraints

for g, $h : \mathbf{R}^n \to \mathbf{R}$, the equality constraint

$$g(x) = h(x)$$

is equivalent to the two inequality constraints

$$g(x) \leq h(x) \text{ and } g(x) \geq h(x),$$

which are equivalent to

$$f_1(x) \leq 0 \text{ and } f_2(x) \leq 0$$

with $f_1(x) = g(x) - h(x)$ and $f_2(x) = h(x) - g(x)$

Feasibility problems

- suppose we only want to
  - find any $x \in \mathbf{R}^n$
  - satisfying $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- this is equivalent to the optimization problem
  - choose $x \in \mathbf{R}^n$
  - to minimize 0
  - subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

Feasibility problems (example)

solving the system of nonlinear equations

$$g_1(x) = h_1(x),\ \ldots,\ g_m(x) = h_m(x)$$

is equivalent to solving the feasibility problem

- find $x \in \mathbf{R}^n$
- subject to $g_i(x) - h_i(x) \leq 0,\ h_i(x) - g(x) \leq 0,\ i = 1, \ldots, m$

Outline

Infeasible problems

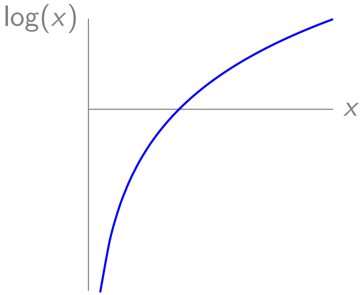
- a problem is **infeasible** if no feasible x exists
- example: minimize $x \in \mathbf{R}$ subject to $x \geq 2$, $x^2 \leq 1$

Unbounded problems

- a problem is **unbounded** if there is a sequence of feasible $x(k)$ s.t.

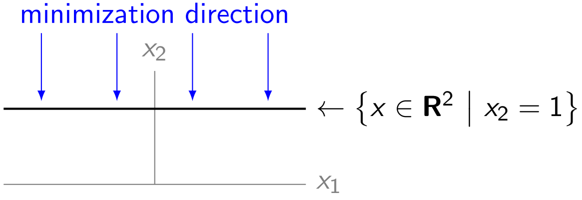
$$f_0(x(k)) \to -\infty \text{ as } k \to \infty$$

example: minimize $\log(x)$ (take $x(1) = 1$, $x(k+1) = x(k)/2$)



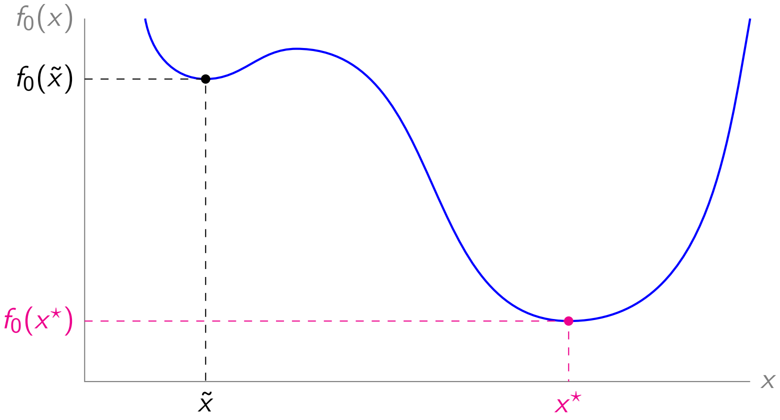
Optimality

- an $x^\star \in \mathbf{R}^n$ is **optimal** (or an **optimizer**) if
  - $x^\star$ is feasible
  - $f_0(x^\star) \leq f_0(x)$ for all feasible x
- infeasible problems have no optimizers
- unbounded problems have no optimizers
- feasible, bounded problems can have multiple optimizers
  - example: choose $x \in \mathbf{R}^2$ to minimize x_2 subject to $x_2 = 1$



Local optimality

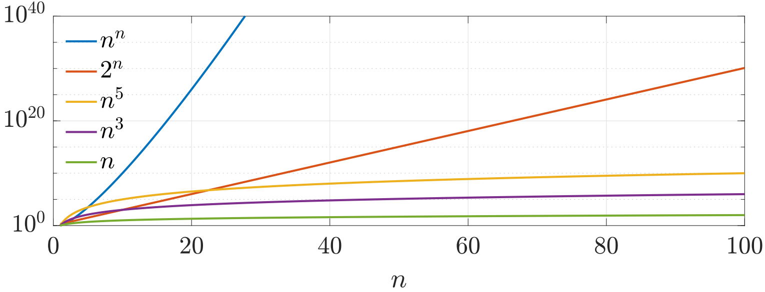
- an $\tilde{x}$ is **locally optimal** (or a **local optimizer**) if
  - $\tilde{x}$ is feasible
  - $f_0(\tilde{x}) \leq f_0(x)$ for all feasible x *in a neighborhood of* $\tilde{x}$
- an unlucky local optimizer $\tilde{x}$ might have $f_0(\tilde{x}) \gg f_0(x^\star)$



Outline

Tractable optimization problems

- few optimization problems can be solved analytically
- but many can be solved numerically
- in general, *global* solve times grow exponentially in n and m
- often, *local* solve times grow only polynomially in n and m



Intractable example: The knapsack problem

- choose $x \in \mathbf{R}^n$
- to maximize $c^\top x$
- subject to $a^\top x \leq b$ and $x_1, \ldots, x_n \in \{0, 1\}$
- given
  - $c \in \mathbf{R}^n$ ('item values')
  - $a \in \mathbf{R}^n$ ('item weights')
  - $b \in \mathbf{R}$ ('knapsack weight limit')
- prove a polynomial-time algorithm? earn \$1 million

Local and global optimization

- a local optimizer $\tilde{x}$
  - can usually be computed efficiently
  - but might be far worse than a global $x^\star$ ($f_0(\tilde{x}) \gg f_0(x^\star)$)
- a global optimizer $x^\star$
  - gives the best feasible performance
  - but might be very slow to compute
- for **convex** problems, *all local optimizers are global optimizers* (more on convexity next lecture)

Least squares

- choose $x \in \mathbf{R}^n$
- to minimize $(Ax - b)^\top (Ax - b)$
- given $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $m \geq n$ (so A is tall)
- idea: no $x \in \mathbf{R}^n$ exactly satisfies all m equations in "$Ax = b$"
- so least squares finds an x with $Ax \approx b$
- analytical solution: $x^\star = (A^\top A)^{-1} A^\top b$ (`A\b` in Matlab)
  (assuming A has linearly independent columns)
- solve time is ~proportional to $n^2 m$

Least squares solution

- for $f(x) = x^\top Px + q^\top x + r$ with $P = P^\top \in \mathbf{R}^{n\times n}$,

$$\nabla f(x) = 2Px + q$$

- least squares has $P = A^\top A$, $q = -2A^\top b$:

$$\begin{aligned}(Ax-b)^\top(Ax-b) &= (x^\top A^\top - b^\top)(Ax-b)\\ &= x^\top A^\top Ax - x^\top A^\top b - b^\top Ax + b^\top b\\ &= x^\top A^\top Ax - 2b^\top Ax + b^\top b\end{aligned}$$

(recalling that $(CD)^\top = D^\top C^\top$ for matrices C and D)

- setting the gradient equal to zero gives

$$2A^\top Ax^\star - 2A^\top b = 0 \iff x^\star = (A^\top A)^{-1}A^\top b$$

provided $A^\top A$ is invertible ($\mathbf{rank}\, A = n$)

One least squares interpretation: Model fitting

- b_i is observation i of a **target** we want to predict (e.g., a community's electricity demand)
- $A_{i1}, \ldots, A_{in}$ are observations i of n predictive **features** (e.g., outdoor temperature, hour, weekday, season, ...)
- $x_1, \ldots, x_n$ are **parameters** in a prediction model
- problem: choose x such that $x_1 A_{i1} + \cdots + x_n A_{in} \approx b_i$ for all i
- the least squares objective

$$(Ax - b)^\top (Ax - b) = \sum_{i=1}^{m} (x_1 A_{i1} + \cdots + x_n A_{in} - b_i)^2$$

penalizes errors between $x_1 A_{i1} + \cdots + x_n A_{in}$ and b_i for all i

Linear programming

- choose $x \in \mathbf{R}^n$
- to minimize $c^\top x$
- subject to $Ax \preceq b$
  (notation: for $y, z \in \mathbf{R}^n$, $y \preceq z$ means $y_1 \leq z_1, \ldots, y_n \leq z_n$)
- given $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$

- no analytical solution, but good algorithms
- solve time is $\sim$proportional to $n^2 m$
- tricks can transform nonlinear problems into linear programs

Linear programming example: Chebyshev approximation

- x, A, b have same interpretations as in least squares example (parameter vector, feature matrix, target vector)
- same goal: choose x such that $x_1 A_{i1} + \cdots + x_n A_{in} \approx b_i$ for all i
- instead of the least squares objective (sum of squared errors)

$$\sum_{i=1}^{m} (x_1 A_{i1} + \cdots + x_n A_{in} - b_i)^2,$$

  use the maximum absolute error

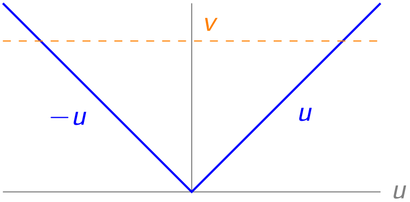
$$\max_{i=1,\ldots,m} |x_1 A_{i1} + \cdots + x_n A_{in} - b_i|$$

- this is not a linear program, but can be transformed into one

Chebyshev approximation as a linear program

- the Chebyshev approximation problem is to
  - choose $x \in \mathbf{R}^n$
  - to minimize $\max_{i=1,\ldots,m} |x_1 A_{i1} + \cdots + x_n A_{in} - b_i|$
- equivalently,
  - choose $(x, y) \in \mathbf{R}^{n+1}$
  - to minimize y
  - subject to $|x_1 A_{i1} + \cdots + x_n A_{in} - b_i| \leq y$, $i = 1, \ldots, m$
- still not a linear program, but closer

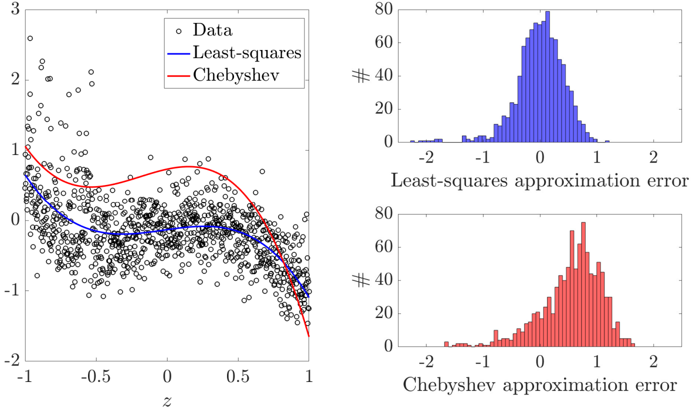
Chebyshev approximation as a linear program (continued)



- for any u, $v \in \mathbf{R}$, $|u| \leq v \iff u \leq v$ and $-u \leq v$
- so an equivalent problem to Chebyshev approximation is to
  - ⋄ choose $(x, y) \in \mathbf{R}^{n+1}$
  - ⋄ to minimize y
  - ⋄ subject to
    $x_1 A_{i1} + \cdots + x_n A_{in} - b_i \leq y$, $i = 1, \ldots, m$
    $-(x_1 A_{i1} + \cdots + x_n A_{in} - b_i) \leq y$, $i = 1, \ldots, m$
- a linear program with $n + 1$ variables and $2m$ constraints

Model fitting example

- noisy data generated from unknown function of z: $b_i = f(z_i)$
- goal: approximate each b_i by cubic, $x_1 + x_2 z_i + x_3 z_i^2 + x_4 z_i^3$
- so $n = 4$ and $A_{ij} = z_i^{j-1}$



Convex optimization

- choose $x \in \mathbf{R}^n$
- to minimize $f_0(x)$
- subject to $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$
- given **convex** $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$

- no analytical solution, but good algorithms
- solve time is ~proportional to $\max\{n^3, n^2 m\}$
- includes least squares, linear programming, and much more

How to use convex optimization

- formulate your problem
- hopefully, recognize it as convex
- otherwise, reformulate or approximate it as convex
- code it in a convex modeling language
  (CVX, CVXPY, Convex.jl, CVXR, ...)
- tell modeling language to pass your problem to a solver
  (SeDuMi, SDPT3, Gurobi, MOSEK, Clarabel, GLPK, ...)
- check solution, tune formulation, repeat until satisfied

Coming soon

- convex sets and functions
- solving convex optimization problems
- DER optimization examples