

# Convex sets and functions

Purdue ME 597, Distributed Energy Resources

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these slides draw on materials by [Stephen Boyd](#) at Stanford

# Outline

Convex sets

Convex functions

Composition rules

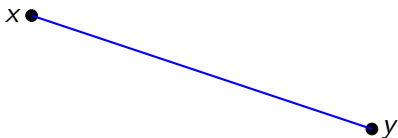
Example functions

# Line segments in $\mathbf{R}^n$

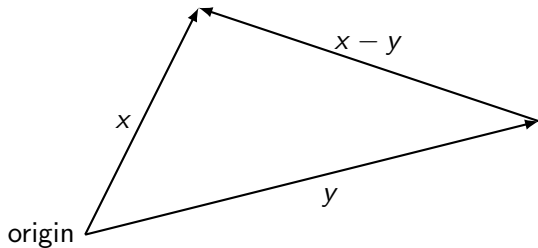
for  $x, y \in \mathbf{R}^n$ ,

$$\{\theta x + (1 - \theta)y \mid \theta \in [0, 1]\}$$

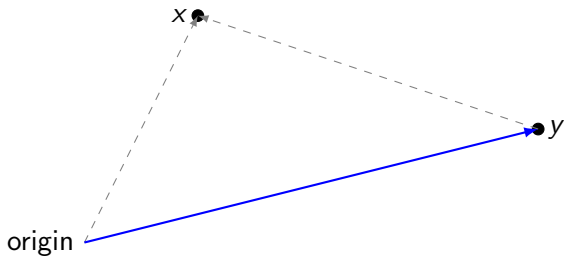
is the line segment connecting  $x$  and  $y$



## Line segments in $\mathbf{R}^n$ (continued)

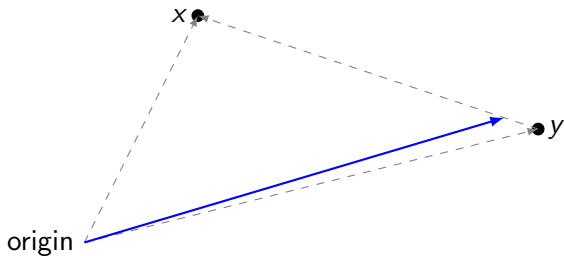


$$\theta x + (1 - \theta)y \text{ with } \theta = 0$$



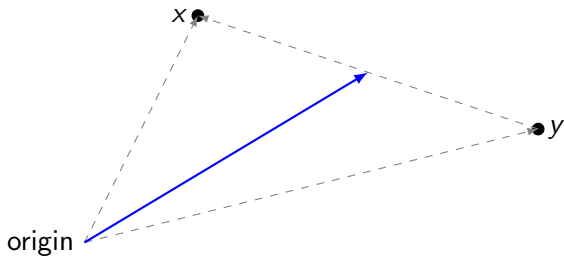
$$0x + (1 - 0)y = y$$

$\theta x + (1 - \theta)y$  with  $\theta = 0.1$



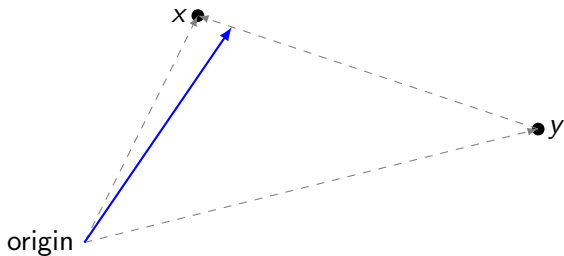
$$0.1x + (1 - 0.1)y = y + 0.1(x - y)$$

$\theta x + (1 - \theta)y$  with  $\theta = 0.5$



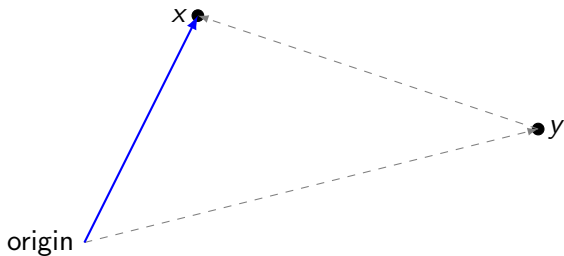
$$0.5x + (1 - 0.5)y = y + 0.5(x - y)$$

$\theta x + (1 - \theta)y$  with  $\theta = 0.9$



$$0.9x + (1 - 0.9)y = y + 0.9(x - y)$$

$\theta x + (1 - \theta)y$  with  $\theta = 1$



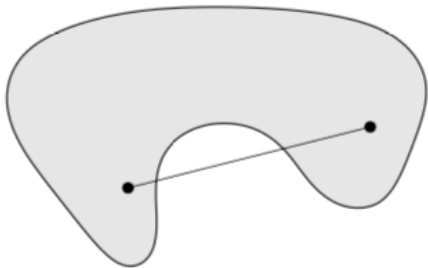
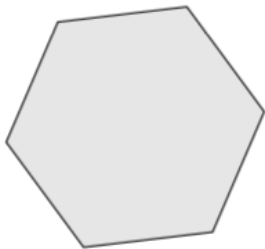
$$1x + (1 - 1)y = x$$

# Convex sets

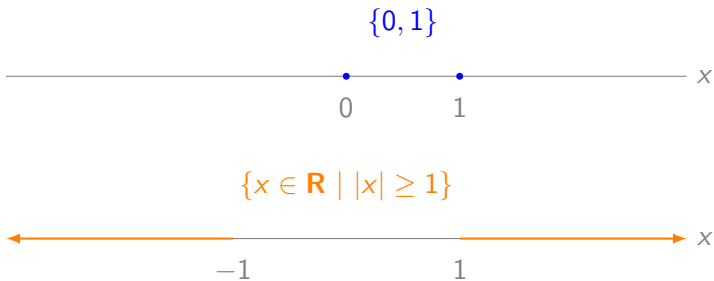
a set  $C \subseteq \mathbf{R}^n$  is **convex** if for all  $x, y \in C$  and  $\theta \in [0, 1]$ ,

$$\theta x + (1 - \theta)y \in C$$

' $C$  contains the entire line segment connecting any two points in  $C$ '



# Nonconvex subsets of $\mathbf{R}$



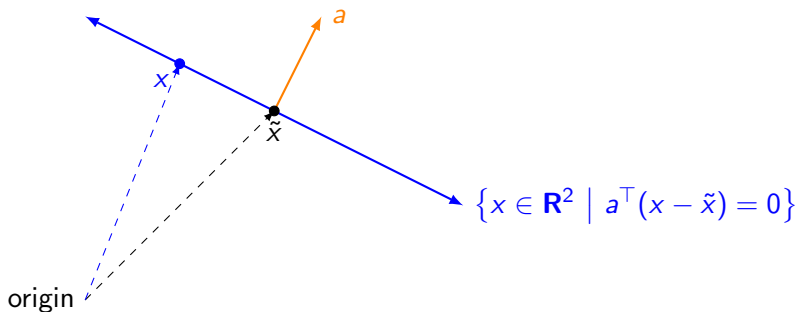
# Hyperplanes

- any  $b \in \mathbf{R}$  and nonzero  $a \in \mathbf{R}^n$  define a hyperplane,

$$\{x \in \mathbf{R}^n \mid a^\top x = b\}$$

- equivalent representation for any  $\tilde{x}$  satisfying  $a^\top \tilde{x} = b$ :

$$\{x \in \mathbf{R}^n \mid a^\top (x - \tilde{x}) = 0\}$$



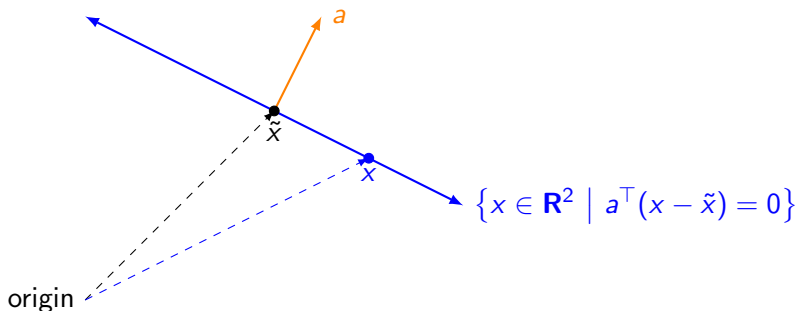
# Hyperplanes

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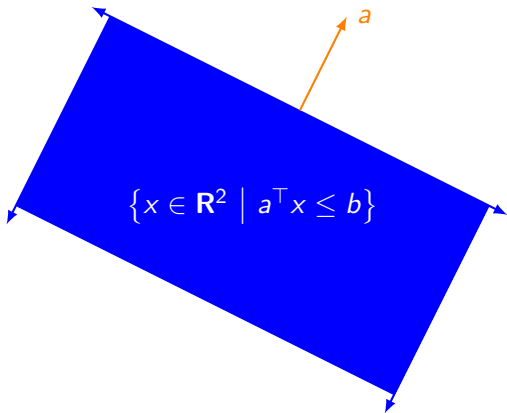
$$\{x \in \mathbf{R}^n \mid a^\top (x - \tilde{x}) = 0\}$$



# Halfspaces

any  $a \neq 0$  and  $b$  (or  $\tilde{x}$  with  $a^\top \tilde{x} = b$ ) define a halfspace,

$$\{x \in \mathbf{R}^n \mid a^\top x \leq b\} = \{x \in \mathbf{R}^n \mid a^\top (x - \tilde{x}) \leq 0\}$$



# Hyperplanes and halfspaces are convex

if  $a^\top x \leq b$  and  $a^\top y \leq b$ , then for any  $\theta \in [0, 1]$ ,

$$\begin{aligned} a^\top(\theta x + (1 - \theta)y) &= \theta a^\top x + (1 - \theta)a^\top y \\ &\leq \theta b + (1 - \theta)b \\ &= b \end{aligned}$$

## Intersections of convex sets are convex

- suppose sets  $C_i \subseteq \mathbf{R}^n$  are convex for  $i = 1, 2, \dots$
- take any  $x, y \in \bigcap_i C_i$   
(this just means that for all  $i$ , both  $x$  and  $y$  are in  $C_i$ )
- each  $C_i$  is convex, so for any  $\theta \in [0, 1]$ ,

$$\theta x + (1 - \theta)y \in C_i$$

- since  $\theta x + (1 - \theta)y \in C_i$  for all  $i$ ,  $\theta x + (1 - \theta)y \in \bigcap_i C_i$

- a **polyhedron** is a set

$$\left\{ x \in \mathbf{R}^n \mid \begin{array}{l} a_i^\top x \leq b_i \text{ for } i = 1, \dots, m \\ c_j^\top x = d_j \text{ for } j = 1, \dots, p \end{array} \right\}$$

of solutions to finitely many linear inequalities and equations

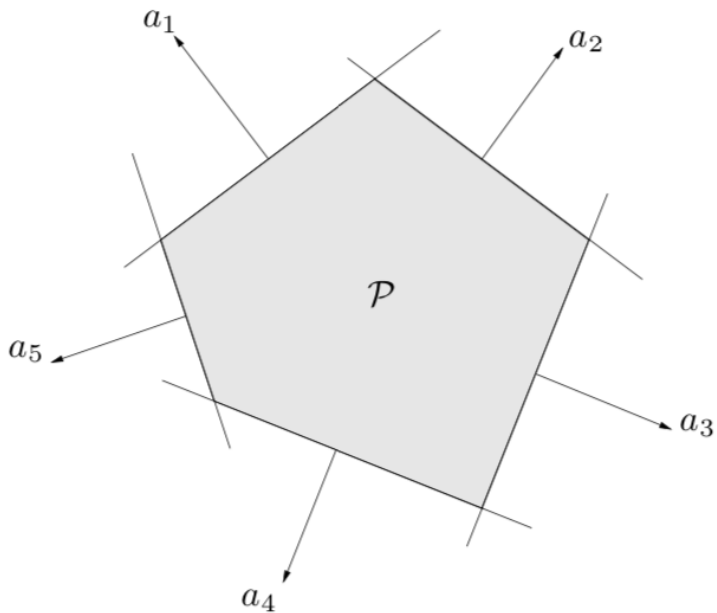
- a polyhedron can be written as

$$\left( \bigcap_{i=1}^m \{x \in \mathbf{R}^n \mid a_i^\top x \leq b_i\} \right) \cap \left( \bigcap_{j=1}^p \{x \in \mathbf{R}^n \mid c_j^\top x = b_j\} \right),$$

the intersection of  $m$  halfspaces and  $p$  hyperplanes

$\implies$  polyhedra are convex

## Polyhedra (continued)



# Outline

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Convex functions

Composition rules

Example functions

- the **domain** of  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is

$$\mathbf{dom} f = \{x \in \mathbf{R}^n \mid f(x) \text{ is defined}\}$$

- example: for  $\log : \mathbf{R} \rightarrow \mathbf{R}$ ,  $\mathbf{dom} \log = \{x \in \mathbf{R} \mid x > 0\}$

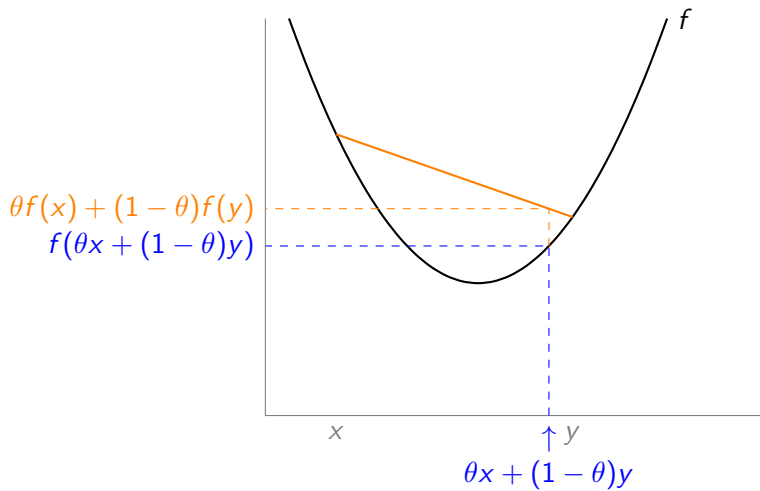
$f : \mathbf{R}^n \rightarrow \mathbf{R}$  is **convex** if

- **dom**  $f$  is convex
- for all  $x, y \in \mathbf{dom} f$  and  $\theta \in [0, 1]$ ,

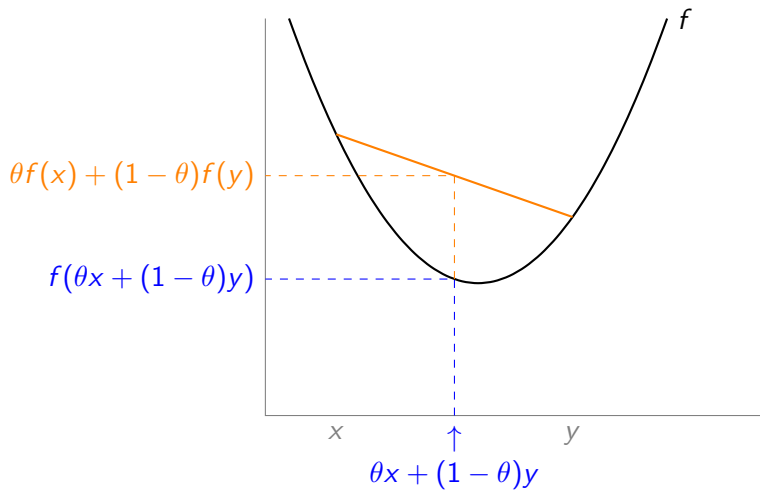
$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

'the line segment from  $(x, f(x))$  to  $(y, f(y))$  lies above the graph of  $f$ '

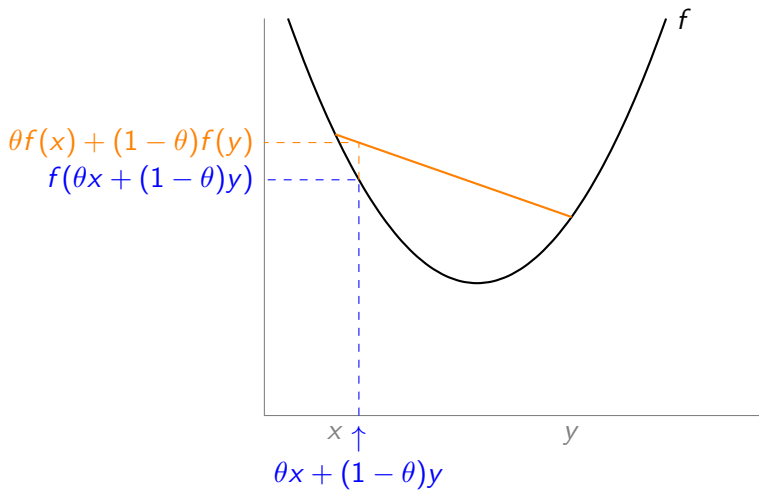
$$\theta = 0.1$$



$$\theta = 0.5$$



$$\theta = 0.9$$



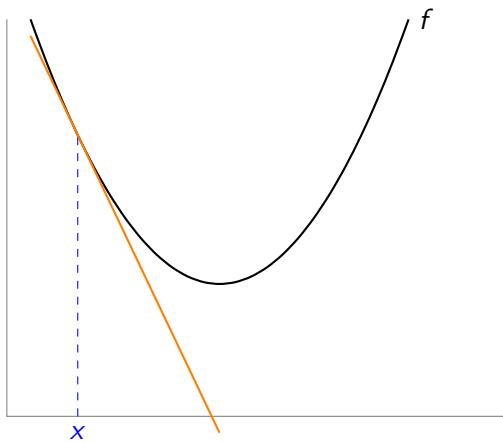
- $S \subseteq \mathbf{R}^n$  is **open** if for all  $x \in S$ ,  $S$  contains a neighborhood of  $x$
- $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is **differentiable** if
  - ◇ **dom**  $f$  is open
  - ◇ the gradient  $\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right)$  exists at each  $x \in \mathbf{dom} f$

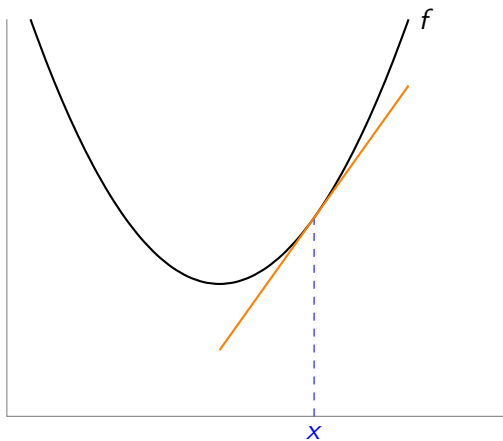
# First-order convexity condition for differentiable functions

- suppose  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is differentiable and  $\mathbf{dom} f$  is convex
- then  $f$  is convex if and only if for all  $x, y \in \mathbf{dom} f$ ,

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

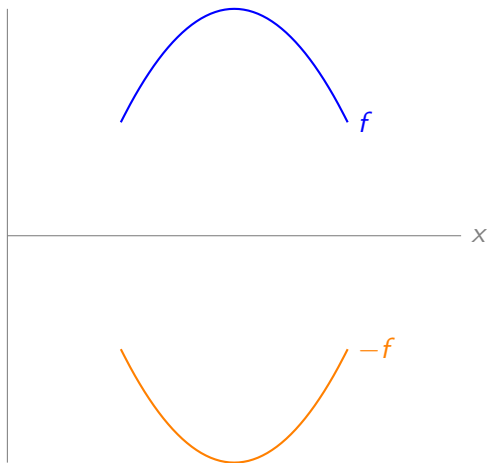
- ‘the linearization of  $f$  at any  $x$  globally underestimates  $f$ ’





# Concave functions

$f : \mathbf{R}^n \rightarrow \mathbf{R}$  is **concave** if  $-f$  is convex



# Affine functions are convex (and concave)

- $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is **affine** if  $f(x) = a^\top x + b$  for some  $a$  and  $b$  (and **linear** if  $f(x) = a^\top x$  for some  $a$ )
- if  $f$  is affine, then  $f$  is convex (and concave):

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= a^\top (\theta x + (1 - \theta)y) + b \\ &= \theta a^\top x + (1 - \theta)a^\top y + b \\ &= \theta(a^\top x + b) + (1 - \theta)(a^\top y + b) \\ &= \theta f(x) + (1 - \theta)f(y) \end{aligned}$$

- conversely, any function that's convex and concave is affine

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Convex functions

**Composition rules**

Example functions

# Monotonicity

- $f : \mathbf{R} \rightarrow \mathbf{R}$  is **nondecreasing** if

$$x \geq y \implies f(x) \geq f(y)$$

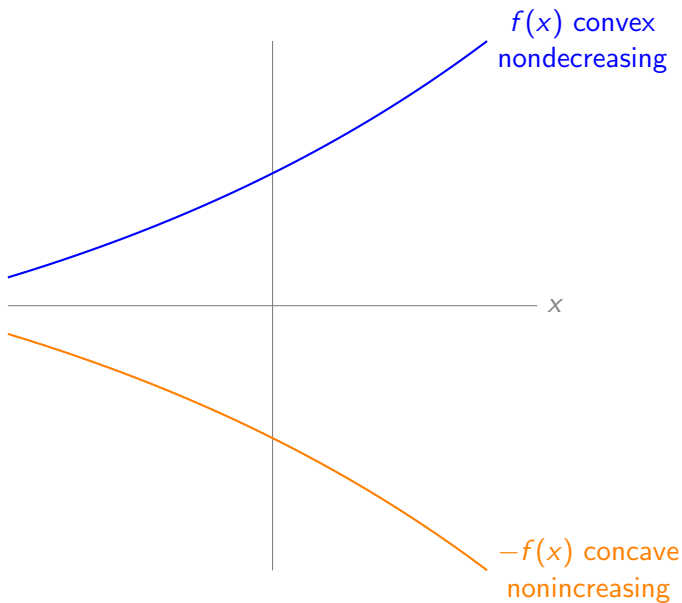
(and **increasing** if  $x > y \implies f(x) > f(y)$ )

- similarly,  $f$  is **nonincreasing** if

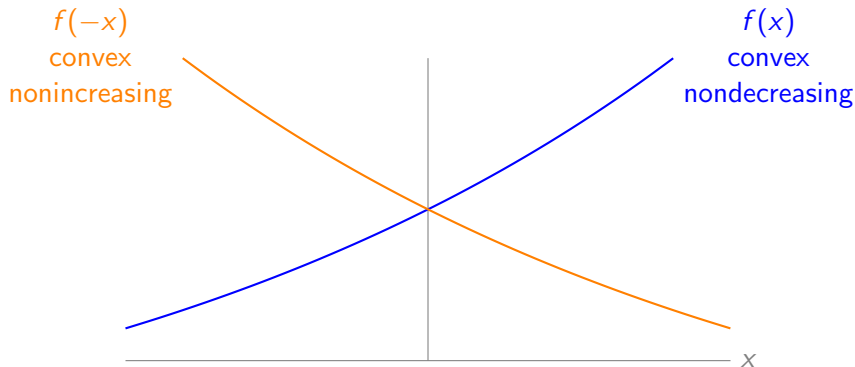
$$x \geq y \implies f(x) \leq f(y)$$

(and **decreasing** if  $x > y \implies f(x) < f(y)$ )

$f(x)$  convex nondec.  $\iff -f(x)$  concave noninc.



$f(x)$  convex nondec.  $\iff f(-x)$  convex noninc.



# The fundamental composition rule

- consider  $h_1, \dots, h_m : \mathbf{R}^n \rightarrow \mathbf{R}$  and convex  $g : \mathbf{R}^m \rightarrow \mathbf{R}$
- define  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  by  $f(x) = g(h_1(x), \dots, h_m(x))$
- $f$  is convex if for each  $i = 1, \dots, m$ ,
  - ◊  $h_i$  is affine, or
  - ◊  $g$  is nondecreasing in argument  $i$  and  $h_i$  is convex, or
  - ◊  $g$  is nonincreasing in argument  $i$  and  $h_i$  is concave
- less precisely but perhaps more memorably,
  - ◊  $\text{CVX}(\text{AFF}) = \text{CVX}$
  - ◊  $\text{CVXND}(\text{CVX}) = \text{CVX}$
  - ◊  $\text{CVXNI}(\text{CCV}) = \text{CVX}$

## Composition rules for concave functions

- consider  $h_1, \dots, h_m : \mathbf{R}^n \rightarrow \mathbf{R}$  and **concave**  $g : \mathbf{R}^m \rightarrow \mathbf{R}$
- define  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  by  $f(x) = g(h_1(x), \dots, h_m(x))$
- $f$  is **concave** if for each  $i = 1, \dots, m$ ,
  - ◇  $h_i$  is affine, or
  - ◇  $g$  is nondecreasing in argument  $i$  and  $h_i$  is **concave**, or
  - ◇  $g$  is nonincreasing in argument  $i$  and  $h_i$  is **convex**

## Useful special cases

- $h_1, h_2$  convex  $\implies h_1 + h_2$  convex
- $h_1$  convex,  $h_2$  concave  $\implies h_1 - h_2$  convex
- $h$  convex, scalar  $\alpha \geq 0 \implies \alpha h$  convex
- $h$  concave, scalar  $\alpha \geq 0 \implies \alpha h$  concave
- $h_i$  convex, scalars  $\alpha_i \geq 0 \implies \alpha_1 h_1 + \dots + \alpha_m h_m$  convex
- $h_1, \dots, h_m$  convex  $\implies \max\{h_1, \dots, h_m\}$  convex

## Composition rules for monotonicity

- consider  $g, h : \mathbf{R} \rightarrow \mathbf{R}$
- define  $f : \mathbf{R} \rightarrow \mathbf{R}$  by  $f(x) = g(h(x))$
- if  $g$  and  $h$  are nondecreasing, then  $f$  is nondecreasing:

$$x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y))$$

- if  $g$  and  $h$  are nonincreasing, then  $f$  is nondecreasing:

$$x \leq y \implies h(x) \geq h(y) \implies g(h(x)) \leq g(h(y))$$

- if  $g$  is nonincreasing and  $h$  is nondecreasing, then  $f$  is nonincreasing:

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- if  $g$  is nondecreasing and  $h$  is nonincreasing, then  $f$  is nonincreasing:

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# Outline

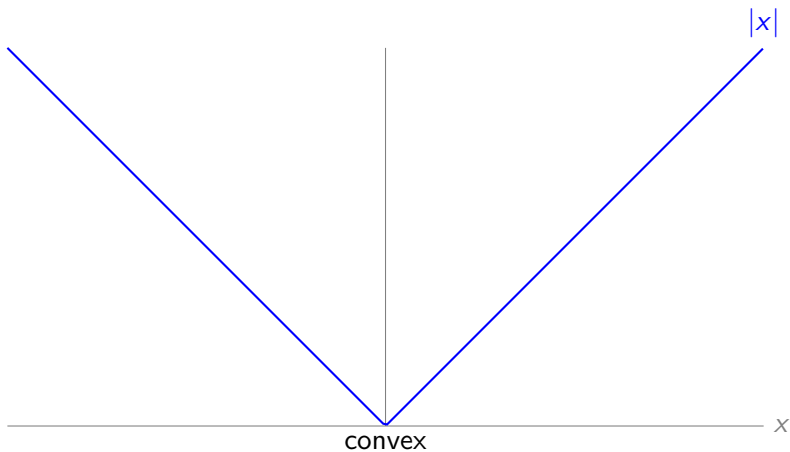
Convex sets

Convex functions

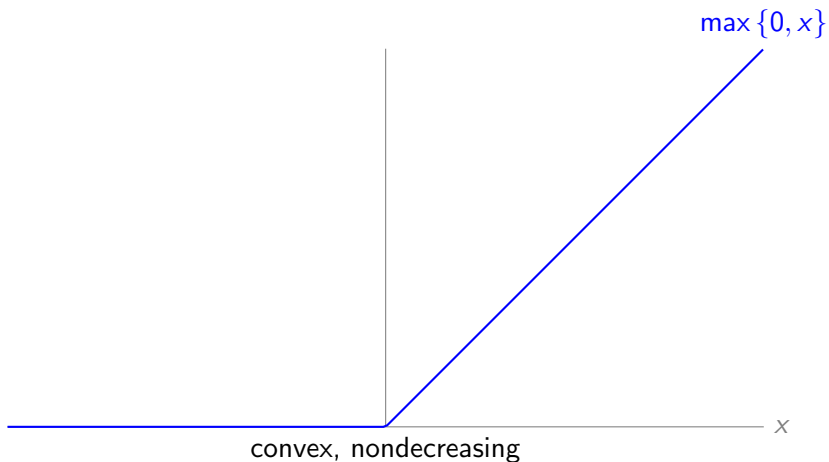
Composition rules

Example functions

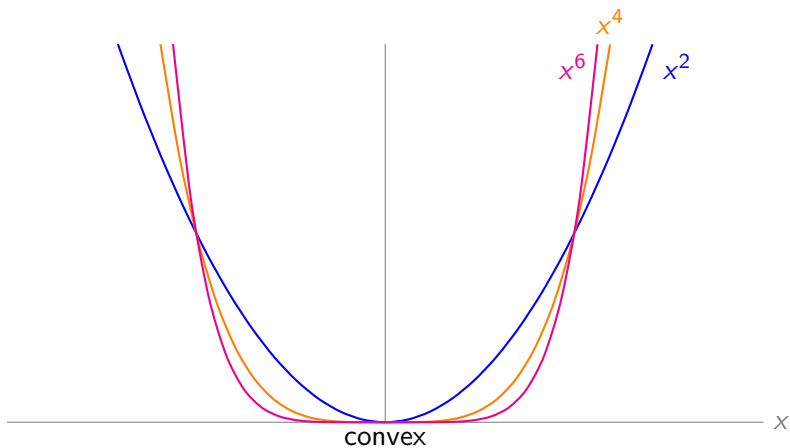
$$f(x) = |x| \text{ with } x \in \mathbf{R}$$



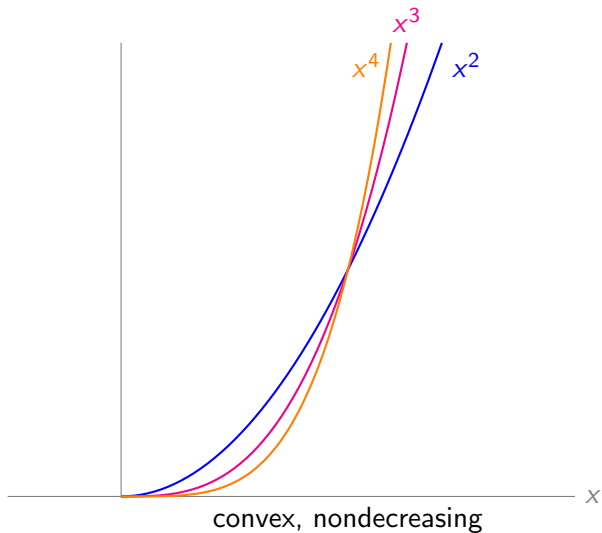
$$f(x) = \max\{0, x\} \text{ with } x \in \mathbf{R}$$



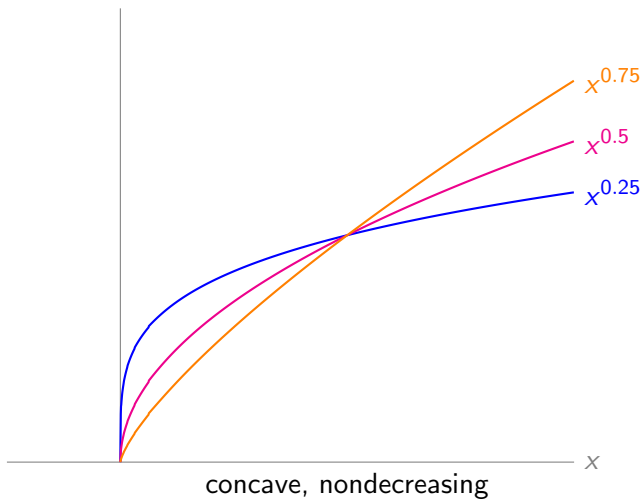
$f(x) = x^p$  with  $x \in \mathbf{R}$  and even, positive  $p$



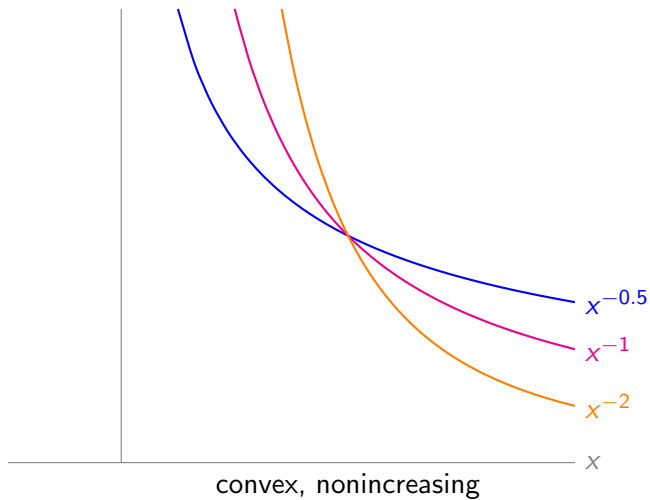
$$f(x) = x^p \text{ with } x \geq 0 \text{ and } p > 1$$



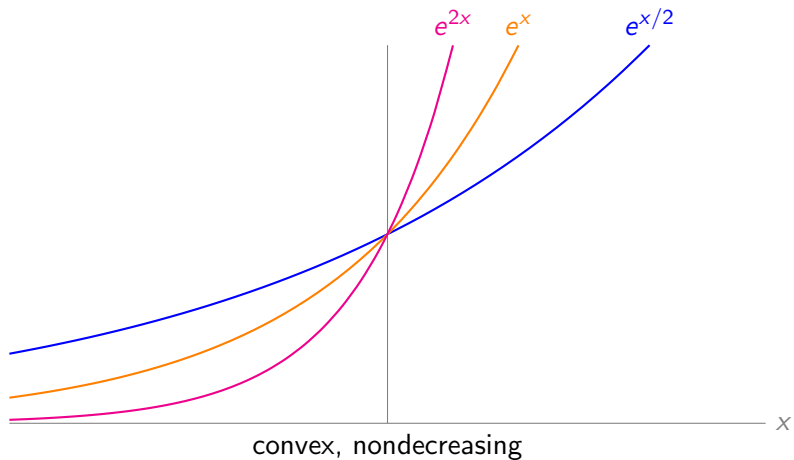
$$f(x) = x^p \text{ with } x \geq 0 \text{ and } p \in (0, 1)$$



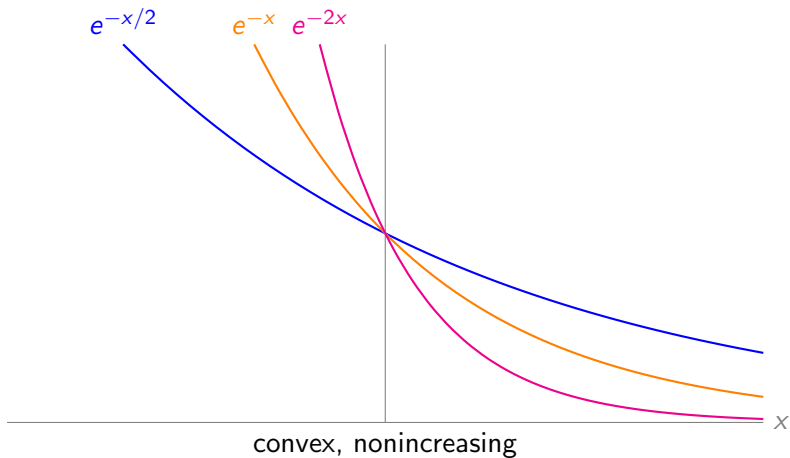
$$f(x) = x^p \text{ with } x > 0 \text{ and } p < 0$$



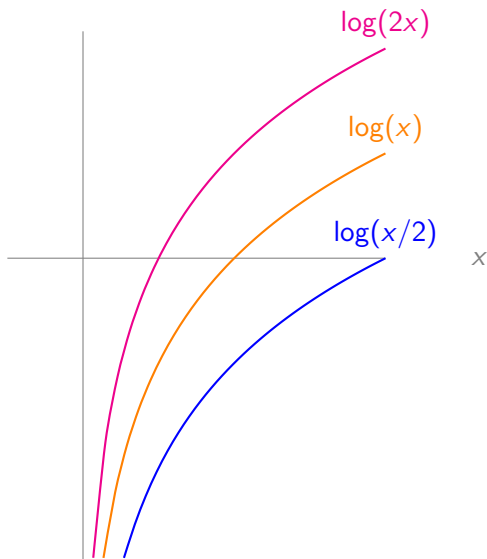
$$f(x) = e^{\alpha x} \text{ with } x \in \mathbf{R}, \alpha \geq 0$$



$$f(x) = e^{\alpha x} \text{ with } x \in \mathbf{R}, \alpha < 0$$

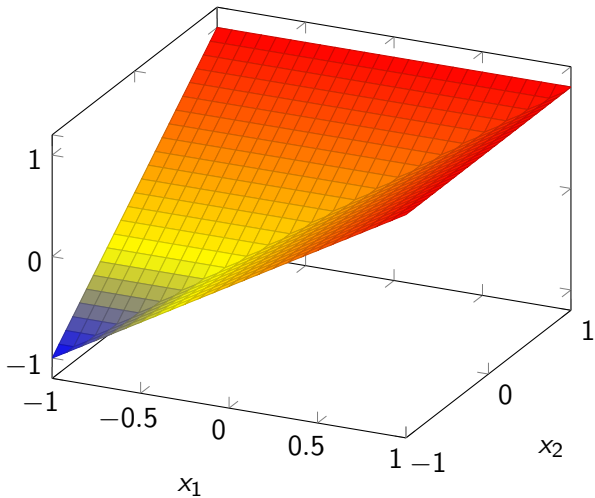


$$f(x) = \log(\alpha x) \text{ with } x > 0, \alpha > 0$$



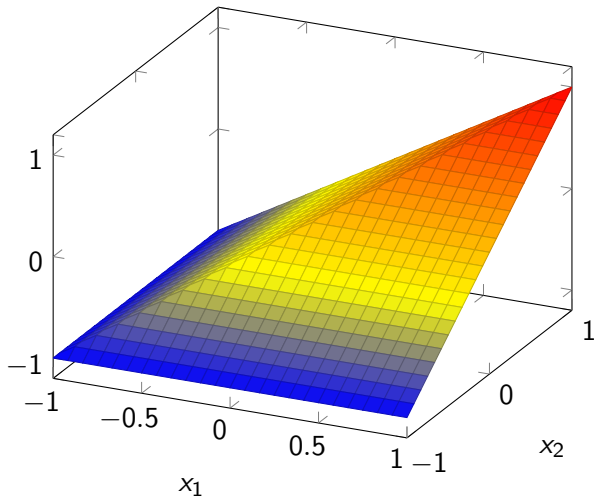
concave, nondecreasing

$$f(x) = \max \{x_1, \dots, x_n\} \text{ with } x \in \mathbf{R}^n$$



convex, (elementwise) nondecreasing

$$f(x) = \min \{x_1, \dots, x_n\} \text{ with } x \in \mathbf{R}^n$$



concave, (elementwise) nondecreasing

- $\| \cdot \| : \mathbf{R}^n \rightarrow \mathbf{R}$  is a **norm** if
  1.  $\|x\| \geq 0$  for all  $x \in \mathbf{R}^n$
  2.  $\|x\| = 0 \iff x = 0$
  3.  $\|\alpha x\| = |\alpha| \|x\|$  for all  $x \in \mathbf{R}^n, \alpha \in \mathbf{R}$
  4.  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in \mathbf{R}^n$
- all norms  $\|x\|$ 
  - ◇ generalize the absolute value  $|x|$  of  $x \in \mathbf{R}$
  - ◇ provide different measures of the length of  $x \in \mathbf{R}^n$   
(or the distance  $\|x - y\|$  between  $x$  and  $y$ )
  - ◇ are convex

# Norm examples

- taxicab or  $l_1$  norm:  $\|x\|_1 = |x_1| + \dots + |x_n|$
- Euclidean or  $l_2$  norm:  $\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$
- Chebyshev or  $l_\infty$  norm:  $\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

