

Convex sets and functions

Purdue ME 597, Distributed Energy Resources

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these slides draw on materials by [Stephen Boyd](#) at Stanford

Outline

Convex sets

Convex functions

Composition rules

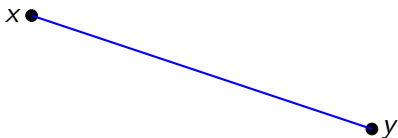
Example functions

Line segments in \mathbf{R}^n

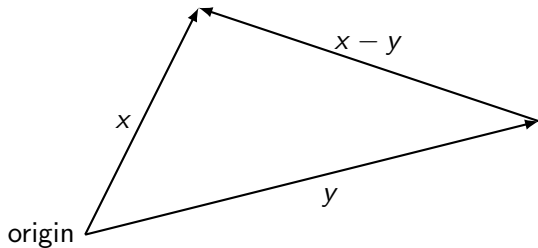
for $x, y \in \mathbf{R}^n$,

$$\{\theta x + (1 - \theta)y \mid \theta \in [0, 1]\}$$

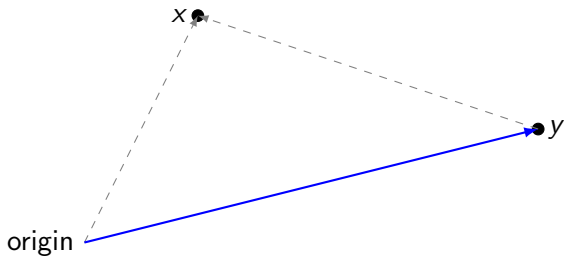
is the line segment connecting x and y



Line segments in \mathbf{R}^n (continued)

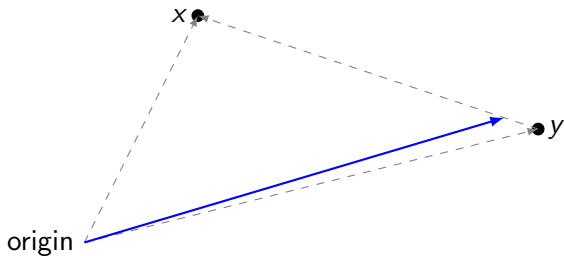


$$\theta x + (1 - \theta)y \text{ with } \theta = 0$$



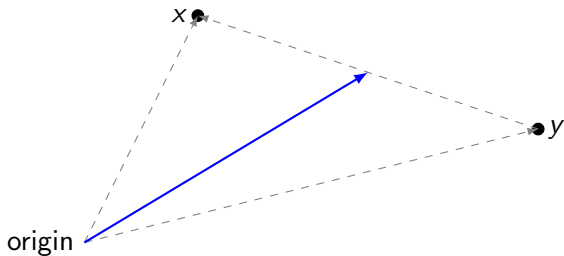
$$0x + (1 - 0)y = y$$

$\theta x + (1 - \theta)y$ with $\theta = 0.1$



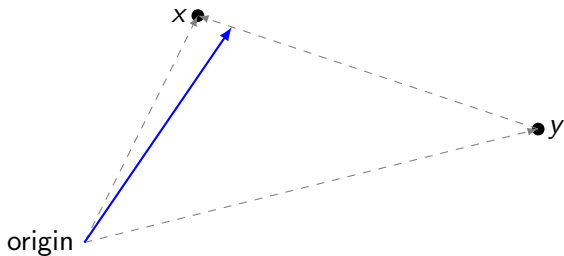
$$0.1x + (1 - 0.1)y = y + 0.1(x - y)$$

$\theta x + (1 - \theta)y$ with $\theta = 0.5$



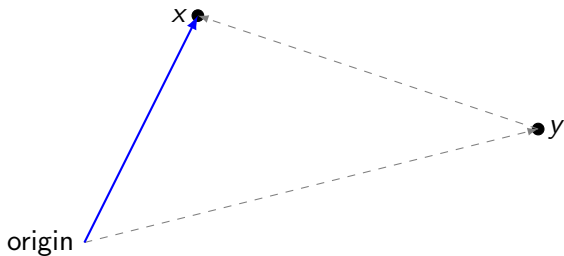
$$0.5x + (1 - 0.5)y = y + 0.5(x - y)$$

$\theta x + (1 - \theta)y$ with $\theta = 0.9$



$$0.9x + (1 - 0.9)y = y + 0.9(x - y)$$

$\theta x + (1 - \theta)y$ with $\theta = 1$

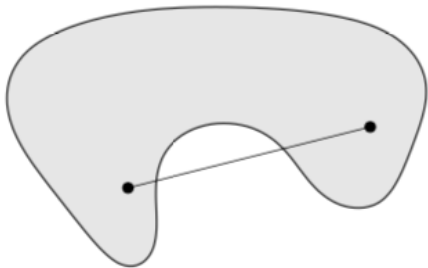
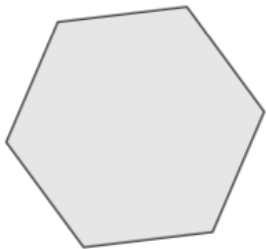


$$1x + (1 - 1)y = x$$

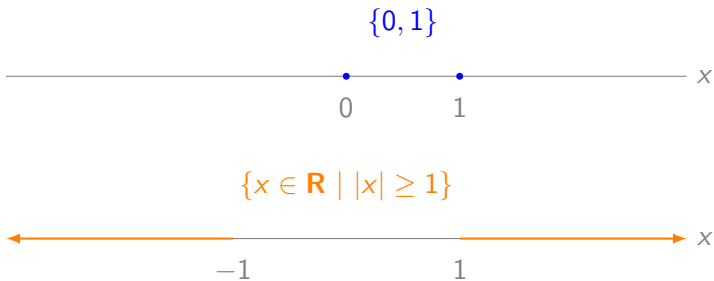
Convex sets

a set $C \subseteq \mathbf{R}^n$ is **convex** if for all $x, y \in C$ and $\theta \in [0, 1]$,

' C contains the entire line segment connecting any two points in C '

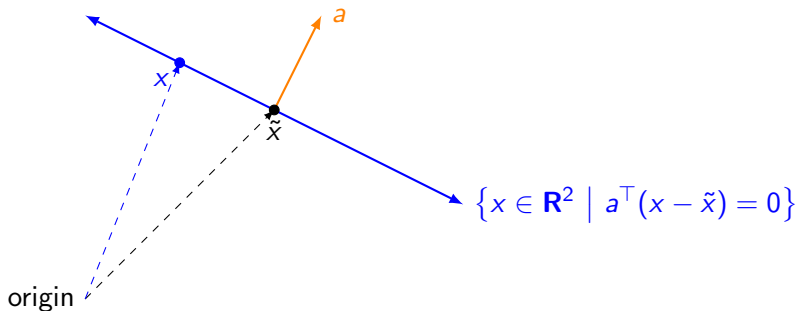


Nonconvex subsets of \mathbf{R}



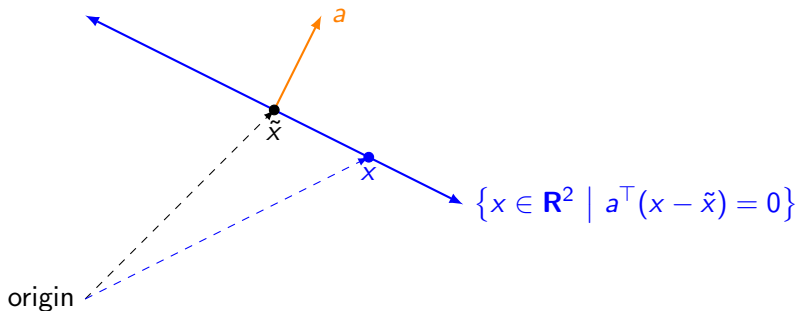
Hyperplanes

- any $b \in \mathbf{R}$ and nonzero $a \in \mathbf{R}^n$ define a hyperplane,
- equivalent representation for any \tilde{x} satisfying $a^\top \tilde{x} = b$:



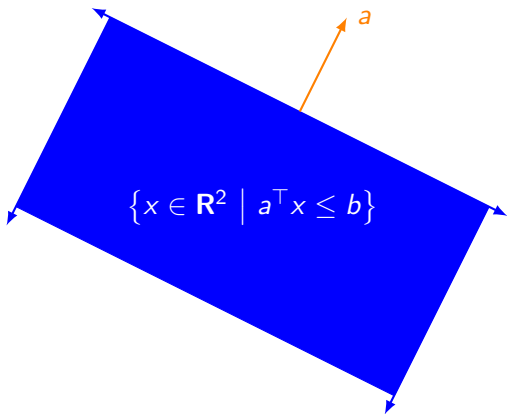
Hyperplanes

- any $b \in \mathbf{R}$ and nonzero $a \in \mathbf{R}^n$ define a hyperplane,
- equivalent representation for any \tilde{x} satisfying $a^\top \tilde{x} = b$:



Halfspaces

any $a \neq 0$ and b (or \tilde{x} with $a^\top \tilde{x} = b$) define a halfspace,



Hyperplanes and halfspaces are convex

if $a^\top x \leq b$ and $a^\top y \leq b$, then for any $\theta \in [0, 1]$,

$$\begin{aligned} a^\top(\theta x + (1 - \theta)y) &= \\ &\leq \\ &= \end{aligned}$$

Intersections of convex sets are convex

- suppose sets $C_i \subseteq \mathbf{R}^n$ are convex for $i = 1, 2, \dots$
- take any $x, y \in \bigcap_i C_i$
(this just means that for all i , both x and y are in C_i)
- each C_i is convex, so for any $\theta \in [0, 1]$,

$$\theta x + (1 - \theta)y \in C_i$$

- since $\theta x + (1 - \theta)y \in C_i$ for all i , $\theta x + (1 - \theta)y \in \bigcap_i C_i$

- a **polyhedron** is a set

of solutions to finitely many linear inequalities and equations

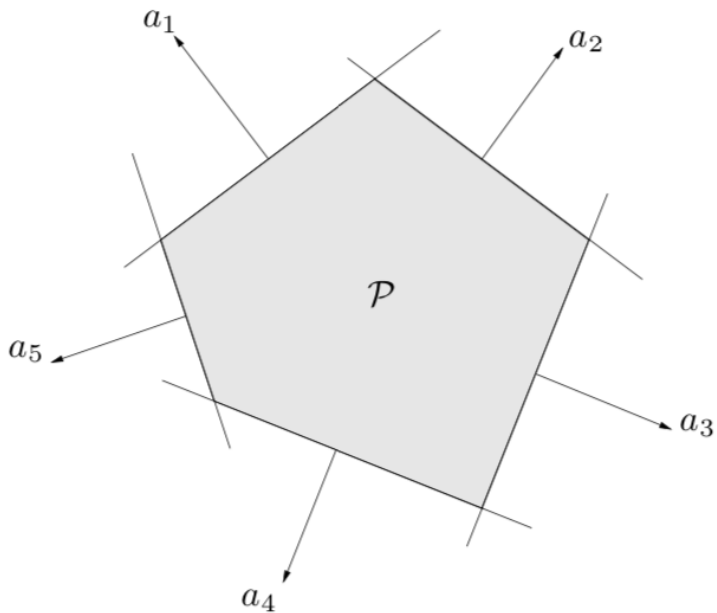
- a polyhedron can be written as

$$\left(\bigcap_{i=1}^m \{x \in \mathbf{R}^n \mid a_i^\top x \leq b_i\} \right) \cap \left(\bigcap_{j=1}^p \{x \in \mathbf{R}^n \mid c_j^\top x = b_j\} \right),$$

the intersection of m halfspaces and p hyperplanes

\implies polyhedra are convex

Polyhedra (continued)



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Example functions

- the **domain** of $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is

$$\mathbf{dom} f = \{x \in \mathbf{R}^n \mid f(x) \text{ is defined}\}$$

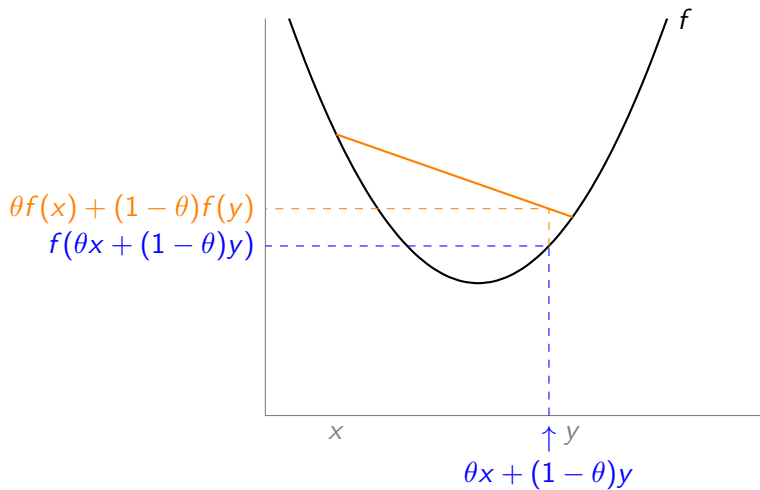
- example: for $\log : \mathbf{R} \rightarrow \mathbf{R}$, $\mathbf{dom} \log = \{x \in \mathbf{R} \mid x > 0\}$

$f : \mathbf{R}^n \rightarrow \mathbf{R}$ is **convex** if

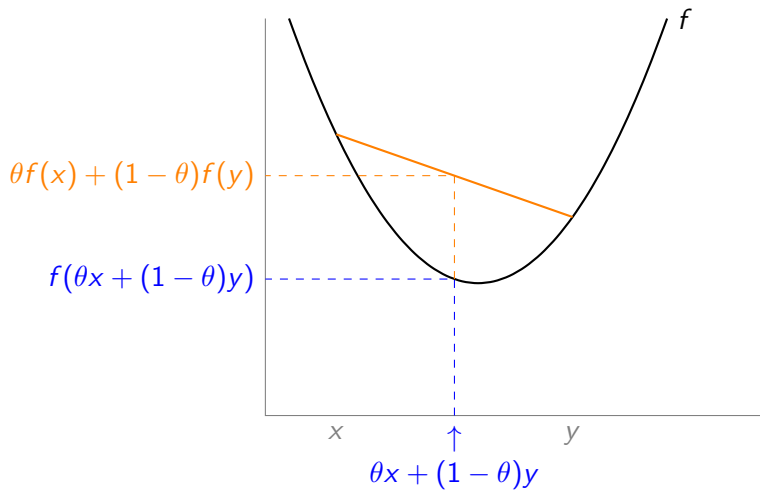
- **dom** f is convex
- for all $x, y \in \mathbf{dom} f$ and $\theta \in [0, 1]$,

'the line segment from $(x, f(x))$ to $(y, f(y))$ lies above the graph of f '

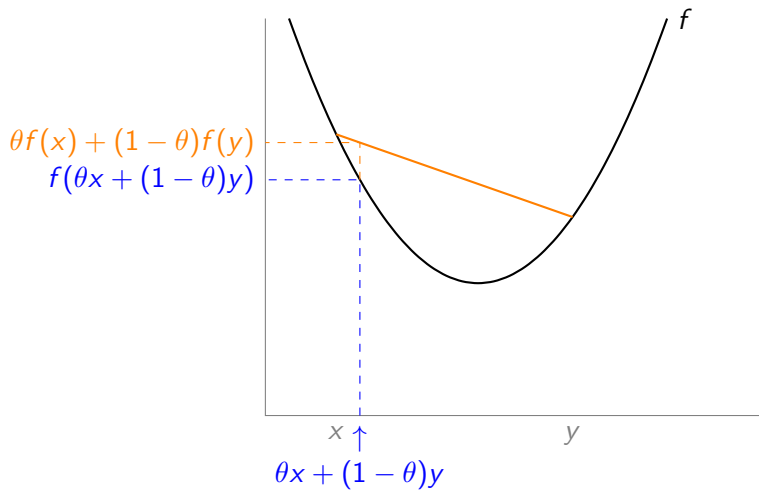
$$\theta = 0.1$$



$$\theta = 0.5$$



$$\theta = 0.9$$

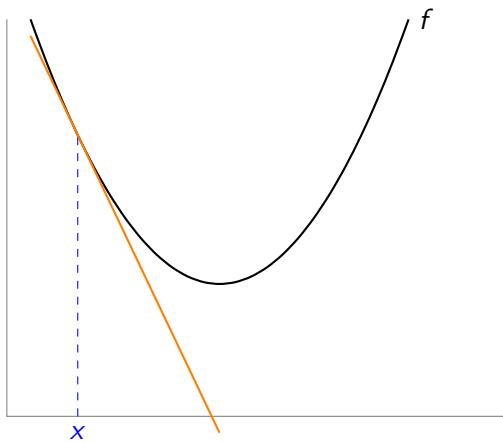


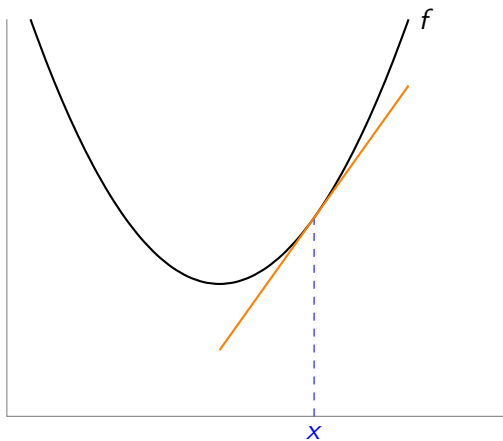
- $S \subseteq \mathbf{R}^n$ is **open** if for all $x \in S$, S contains a neighborhood of x
- $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is **differentiable** if
 - ◇ **dom** f is open
 - ◇ the gradient $\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_n} \right)$ exists at each $x \in \mathbf{dom} f$

First-order convexity condition for differentiable functions

- suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is differentiable and $\mathbf{dom} f$ is convex
- then f is convex if and only if for all $x, y \in \mathbf{dom} f$,

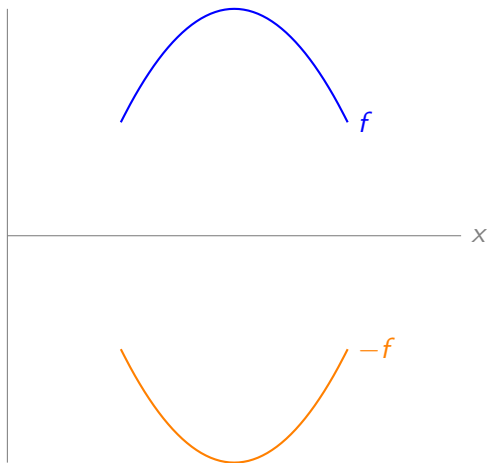
- ‘the linearization of f at any x globally underestimates f ’





Concave functions

$f : \mathbf{R}^n \rightarrow \mathbf{R}$ is **concave** if $-f$ is convex



Affine functions are convex (and concave)

- $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is **affine** if $f(x) = a^\top x + b$ for some a and b (and **linear** if $f(x) = a^\top x$ for some a)
- if f is affine, then f is convex (and concave):

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= \\ &= \\ &= \\ &= \end{aligned}$$

- conversely, any function that's convex and concave is affine

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Monotonicity

- $f : \mathbf{R} \rightarrow \mathbf{R}$ is **nondecreasing** if

$$x \geq y \implies f(x) \geq f(y)$$

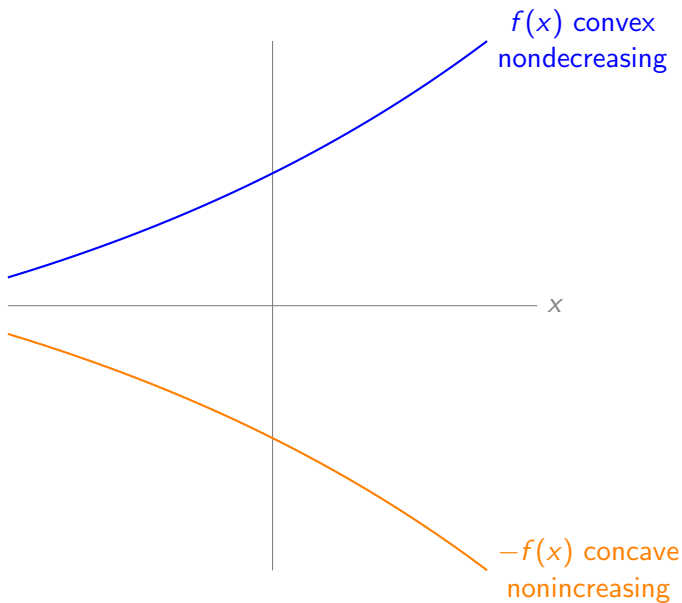
(and **increasing** if $x > y \implies f(x) > f(y)$)

- similarly, f is **nonincreasing** if

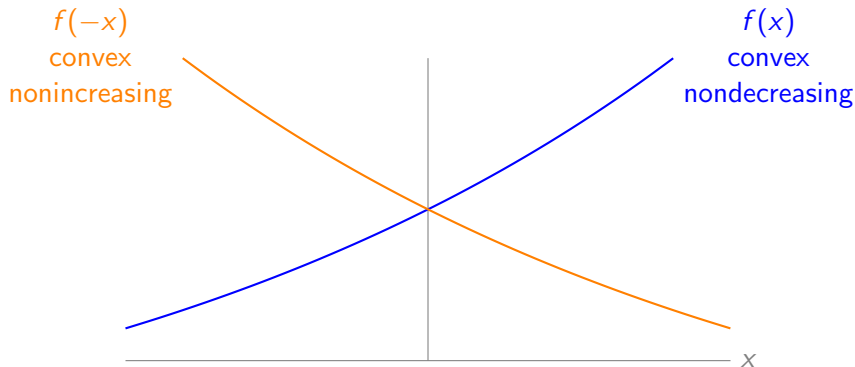
$$x \geq y \implies f(x) \leq f(y)$$

(and **decreasing** if $x > y \implies f(x) < f(y)$)

$f(x)$ convex nondec. $\iff -f(x)$ concave noninc.



$f(x)$ convex nondec. $\iff f(-x)$ convex noninc.



The fundamental composition rule

- consider $h_1, \dots, h_m : \mathbf{R}^n \rightarrow \mathbf{R}$ and convex $g : \mathbf{R}^m \rightarrow \mathbf{R}$
- define $f : \mathbf{R}^n \rightarrow \mathbf{R}$ by $f(x) = g(h_1(x), \dots, h_m(x))$
- f is convex if for each $i = 1, \dots, m$,
 - ◊ h_i is affine, or
 - ◊ g is nondecreasing in argument i and h_i is convex, or
 - ◊ g is nonincreasing in argument i and h_i is concave
- less precisely but perhaps more memorably,
 - ◊ $\text{CVX}(\text{AFF}) = \text{CVX}$
 - ◊ $\text{CVXND}(\text{CVX}) = \text{CVX}$
 - ◊ $\text{CVXNI}(\text{CCV}) = \text{CVX}$

Useful special cases

- h_1, h_2 convex $\implies h_1 + h_2$ convex
- h_1 convex, h_2 concave $\implies h_1 - h_2$ convex
- h convex, scalar $\alpha \geq 0 \implies \alpha h$ convex
- h concave, scalar $\alpha \geq 0 \implies \alpha h$ concave
- h_i convex, scalars $\alpha_i \geq 0 \implies \alpha_1 h_1 + \dots + \alpha_m h_m$ convex
- h_1, \dots, h_m convex $\implies \max\{h_1, \dots, h_m\}$ convex

Outline

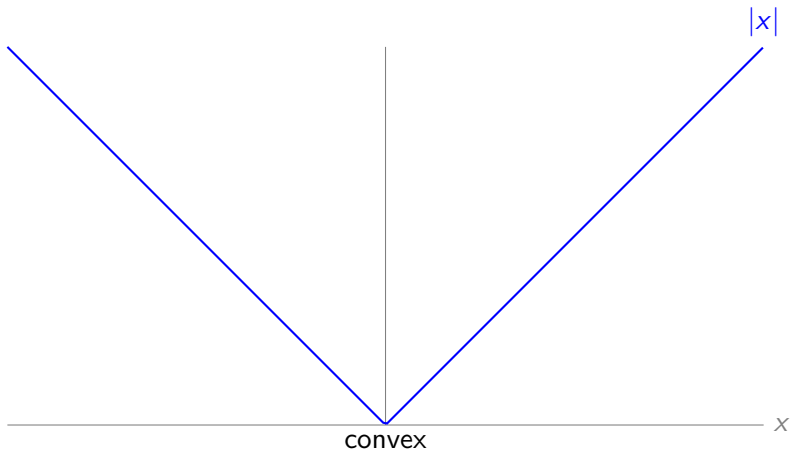
Convex sets

Convex functions

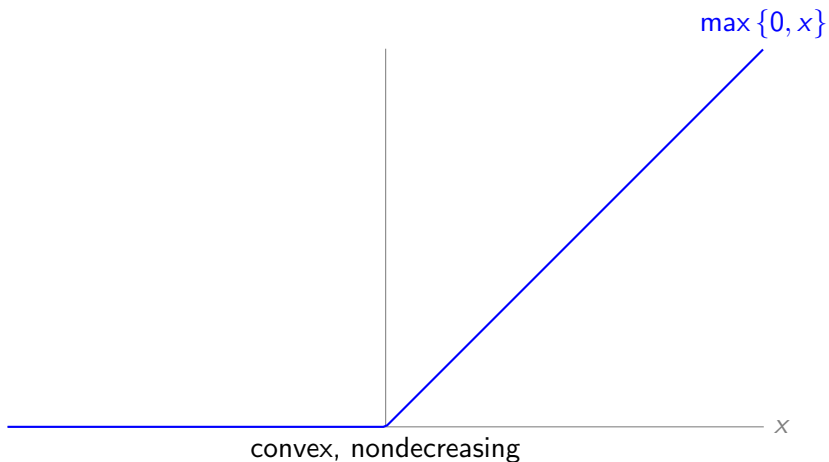
Composition rules

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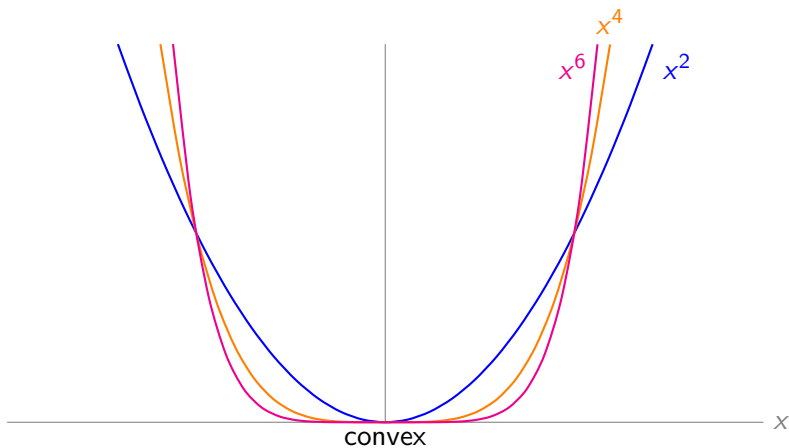
$$f(x) = |x| \text{ with } x \in \mathbf{R}$$



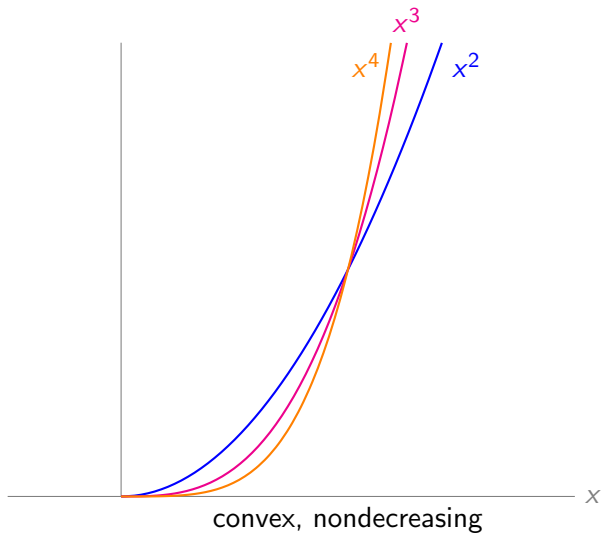
$$f(x) = \max\{0, x\} \text{ with } x \in \mathbf{R}$$



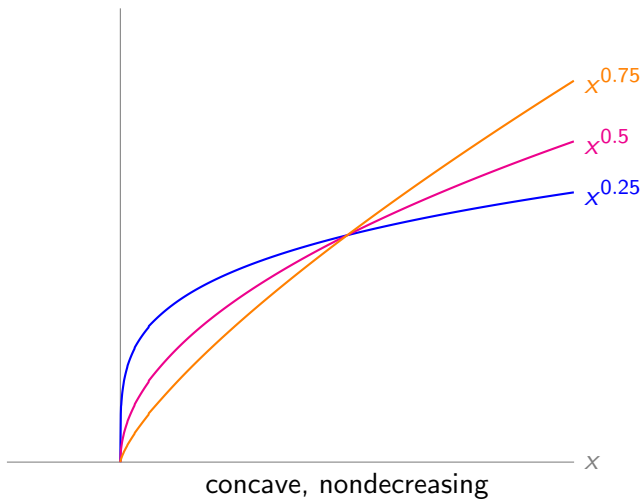
$f(x) = x^p$ with $x \in \mathbf{R}$ and even, positive p



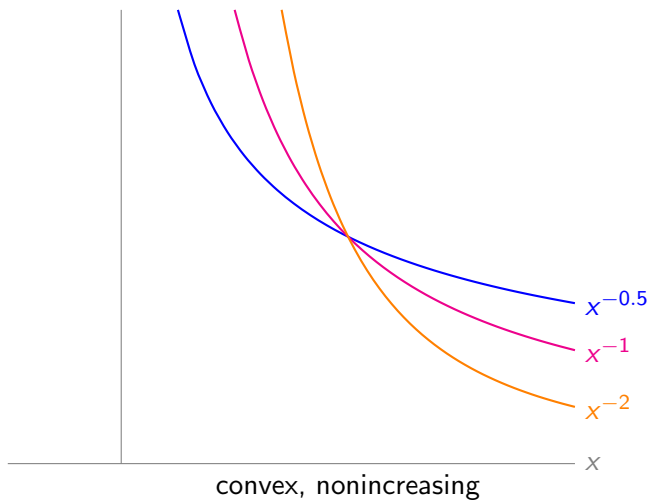
$$f(x) = x^p \text{ with } x \geq 0 \text{ and } p > 1$$



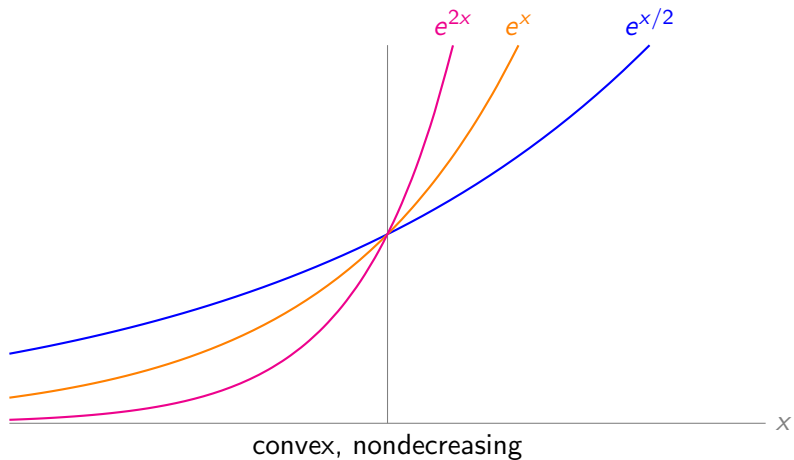
$$f(x) = x^p \text{ with } x \geq 0 \text{ and } p \in (0, 1)$$



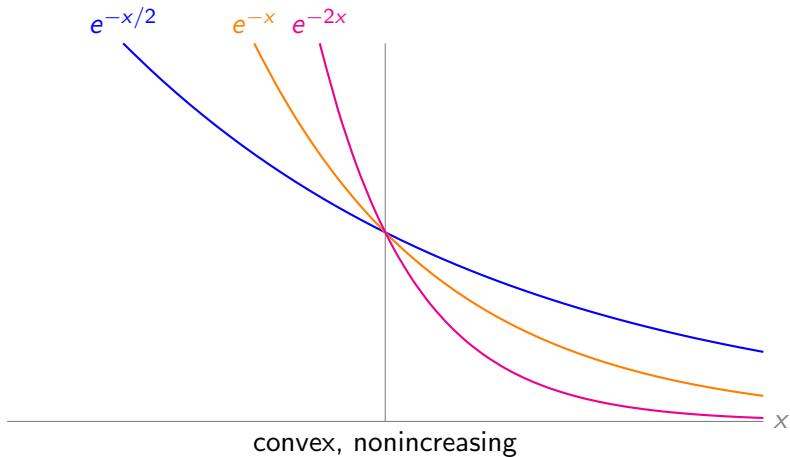
$$f(x) = x^p \text{ with } x > 0 \text{ and } p < 0$$



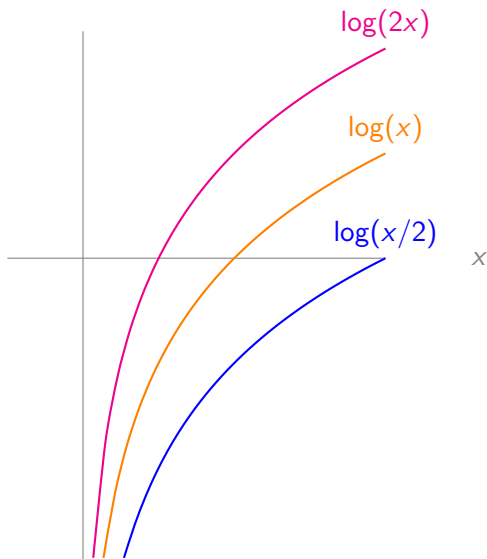
$$f(x) = e^{\alpha x} \text{ with } x \in \mathbf{R}, \alpha \geq 0$$



$$f(x) = e^{\alpha x} \text{ with } x \in \mathbf{R}, \alpha < 0$$

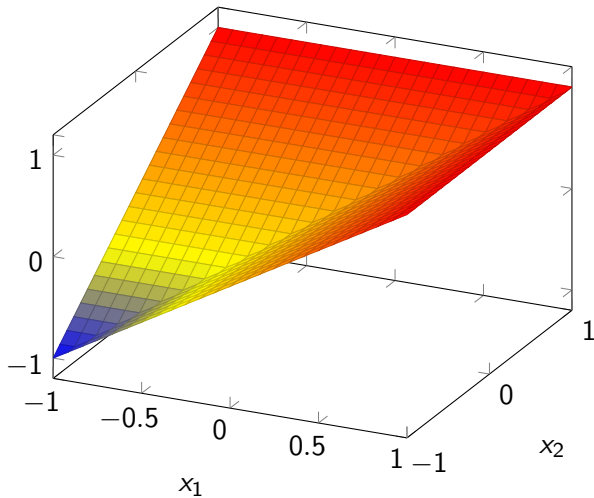


$$f(x) = \log(\alpha x) \text{ with } x > 0, \alpha > 0$$



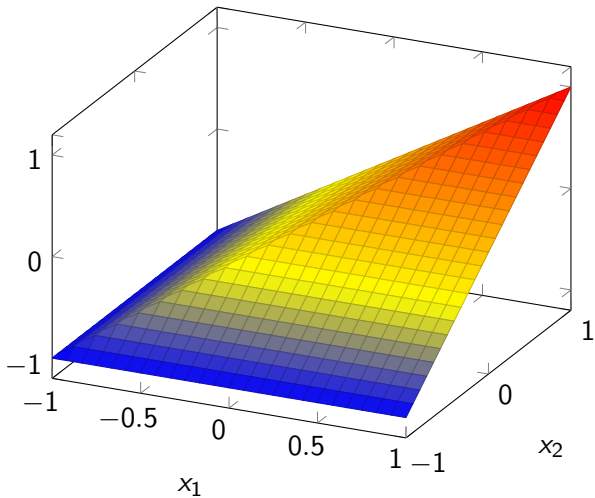
concave, nondecreasing

$$f(x) = \max \{x_1, \dots, x_n\} \text{ with } x \in \mathbf{R}^n$$



convex, (elementwise) nondecreasing

$$f(x) = \min \{x_1, \dots, x_n\} \text{ with } x \in \mathbf{R}^n$$



concave, (elementwise) nondecreasing

- $\| \cdot \| : \mathbf{R}^n \rightarrow \mathbf{R}$ is a **norm** if
 - 1.
 - 2.
 - 3.
 - 4.
- all norms $\|x\|$
 - ◇ generalize the absolute value $|x|$ of $x \in \mathbf{R}$
 - ◇ provide different measures of the length of $x \in \mathbf{R}^n$
(or the distance $\|x - y\|$ between x and y)
 - ◇ are convex

Norm examples

- taxicab or l_1 norm:
- Euclidean or l_2 norm:
- Chebyshev or l_∞ norm:

